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On uniqueness of equilibrium prices in a Bayesian Assignment Game

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Abstract

In the Assignment Game [Shapley and Shubik, 1971], most solution concepts yield a multiplicity of solutions. We study the Assignment game in a Bayesian environment where neither buyers nor sellers know the valuation of other players, and derive conditions on the distribution of valuations to guarantee the uniqueness of equilibrium. Also, we provide a closed-form solution when valuations follow an exponential distribution.

KEYWORDS: Assignment Game, Uniqueness, Newton's Method, Contraction mapping

1 Introduction

In the Assignment Game, introduced by Shapley and Shubik in [1971], the literature focuses on analyzing the conditions under which a competitive equilibrium does exist and the mechanisms to compute them. These studies present characterizations of such equilibria based on core and efficiency features [Bikhchandani and Mamer, 1997, Alaei et al., 2016]. This approach establishes that multiple equilibria may prevail in the Assignment Game, and does not propose refinements to select a particular equilibrium. Our objective, in contrast, is to look at conditions under which comparative statics naturally develop in the assignment games. Explicitly, our approach guarantees uniqueness of equilibrium and a closed form solution of equilibrium pricing strategies.

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We analyze the set of equilibria in the Assignment Game in a Bayesian framework where buyers and sellers have private information. We assume that sellers have different valuations of their goods, and do not know the valuation of other sellers. We consider a variation of the Assignment Game as a two-stage game. In the first stage, nature draws the valuation of each agent, namely over the good they own for sellers and all goods for buyers. At stage two, sellers simultaneously set prices. The final allocation is as follows: buyers rank all goods concerning their surplus and goods are assigned to one of the buyers for whom the surplus is maximal.

In our setting, equilibrium prices are characterized by the inverse hazard rate function of buyers valuations distribution. Our main result establishes a sufficient condition for the existence of a unique price vector at equilibrium; specifically it requires the inverse hazard rate to be a contraction. Geometrically, this means that the inverse hazard rate behaves similarly to a constant function, i.e., we can say that the tail distribution is almost a multiple of the density function. The condition, however, is not necessary since uniqueness is also guaranteed when valuations are uniformly distributed, although their associated inverse hazard rate is not a contraction.

As far as we know, ours is the first approach establishing the uniqueness of equilibrium prices in the Assignment Game. In contrast, the multiplicity of competitive equilibria and core allocations are pervasive in the Assignment Game [Shapley and Scarf, 1971]. Assuming that agents cannot have more than one indivisible good, Quinzii [1982], and Kaneko and Yamamoto [1986] show that the core of the economy is non-empty, and not necessarily unique. Also, Quinzii analyses the conditions under which the core allocations coincide with competitive equilibrium allocations. Similarly, Demange [1984] proves the existence of at least one competitive equilibrium, which is not always unique, in a model with externalities. A generalization of the assignment game is made by [Scarf, 1994] through the analysis of the production of indivisible goods. He emphasizes the problems of performing comparative statistics when indivisibilities cause the failure of competitive prices. Alkan, Demange and Gale [1991] show that the set of equilibria have a lattice structure in the Assignment game, and the set of efficient and envy-free allocations is non-empty. Even in the presence of the multiplicity of fair assignments, they show that it is possible to do some comparative statistics when money increases. In similar models where the multiplicity of fair allocations prevail, Svensson [2009] characterizes the set of fair allocation rules that are strategy-proof, and Tadenuma and Thomson [1991] study when fair allocations satisfy consistency.

Our paper is closely related to the literature that analyses the characteristics of particular classes of games to guarantee the uniqueness of Nash equilibrium like [Green and Laffont, 1984, Papavassilopoulos and Cruz, 1979, Aminof et al., 2016, Kolstad and Mathiesen, 1987]. Hence, we analyse the features of the sellers' expected utility that guarantee a unique price in the Bayesian Assignment Game that we analyse. By assuming statistical independence between distributions of valuations, the first order conditions state that the direct effect by increasing the price is equivalent to the indirect effect provided by the probability of not selling the good. It implies that seller's best responses are implicitly determined by an equation in term of a buyer's inverse hazard rate. Considering that buyers' valuations are uniformly distributed, the previous equation is linear, which induces a unique equilibrium price. In the general case, we observe that each good equilibrium prices are independent of other goods, and they are the fixed points of the inverse hazard function of some buyer who gets the largest surplus by buying it. So, assuming a contraction inverse hazard rate function is sufficient to guarantee the existence and uniqueness of equilibrium prices by the Banach Fixed-Point Theorem. The sufficient condition is tight; we discuss variations under which the condition does not hold. We present an example where a non-contracting inverse hazard rate, non-related to the uniform distribution, induces a high-degree polynomial equation to compute the sellers' best responses. The use of the Contraction Map Theorem in game theory is not new; Long and Soubeyran [2000], and Ceparano and Quatieri [2017] also use this approach to guarantee a unique Nash equilibrium in Cournot games and weighted potential games, respectively.

When valuations are exponentially distributed, the inverse hazard rate is constant and satisfies the contraction property. Hence, our main result guarantees the existence of a unique equilibrium price vector in this case. Even more, the exponential distribution allows us to get a closed form solution for the equilibrium prices and perform some natural comparative statistics. Unsurprisingly, the relation between the price at equilibrium and its corresponding seller valuation is positive; while the price decreases when the parameter of the exponential distribution increases. When the parameter of the exponential distribution is interpreted as the average time required to buy a good, this last result means that the more buyers are in a hurry, the higher the price.

Assuming exponential distribution in our model echoes empirical works in the real estate market [Merlo et al., 2015]. Trippi [1977], Baryla and Ztanpano [1995], and Tsai [2010] suggest that buying a house is an exponentially distributed event, in the sense that it depends on finding a buyer who is willing to pay a certain price. Furthermore, Horowitz (1992) relates the parameter of the exponential function with the number of sellers: for increasing the searching costs, the higher the number of houses on the market, the longer buyers needs to know and compare all available options [Horowitz, 1992, Merlo and Ortalo-Magné, 2004]. Thus, they establish a positive relationship between the exponential parameter's inverse and the number of sellers. Adding this assumption in our model implies that prices increase when the number of sellers increases.

Markets where prices increase with the number of sellers arise in other settings. From a theoretical point of view, Rosenthal [1980] studies a model where sellers have monopolistic power over a fraction of buyers (captive consumers), production costs are zero, and the preferences of non-captive consumers are unknown. In this setting, prices increase when the number of sellers increases because the probability to get non-captive buyers decreases and sellers focus on differentiating their good. Bagwell and Lee [2014] characterize this behaviour in a game where sellers' production costs are different and buyers' valuations are private information. This result is supported by empirical evidence of markets where buyers make decisions based on sellers' reputation and search time [Gill and Thanassoulis, 2015, Head et al., 2014].

Our results are robust to the cases where agents preferences are exponentially distributed but not identically; we present an example where parameters are different between buyers and other where their valuations overlap. Finally, our analyses does not restrict to symmetric equilibrium.

The paper is organized as follows. Section 2 presents the model and the two-stage game. Section 3 analyses the set of equilibria, and the sufficient condition over the valuation distributions to guarantee a unique price vector at equilibrium. In Section 4, we show that valuations exponentially distributed satisfy the conditions for the existence of a single price, which allows computing a closed form for it. Also, we perform some comparative statics. Conclusions are presented in Section 5.

2 The Model

2.1 Buyers and Sellers

We consider a market with indivisible goods, money, and two disjoint sets of agents: a set of sellers, S, and a set of buyers, B. Let r be a generic agent in $S \cup B$. Money is a perfectly divisible good $\omega \in \mathbb{R}$ that agents use to pay the bill. We assume that each agent initially has a certain amount of money $\omega_r \in \mathbb{R}_+$. Also, all sellers initially own one and only one indivisible good, and buyers initially do not own any good. We use $\varnothing$ whenever an agent does not own any good.

Let S be the set of m sellers, we use s_j to denote a generic seller with $j = 1, 2, \ldots, m$. Since each seller s_j initially owns an indivisible good, we identify this good with s_j to avoid extra notation. By simplicity, s_j initially owns an amount of money $\omega_j = 0$. So, the initial endowment of s_j is the money/indivisible good basket $(0, s_j)$. Also, seller s_j has a valuation (type) $v_j \in \mathbb{R}$ of her good. Let V_j be the set of all possible types of seller s_j. We consider that seller s_j has a preference relation over baskets $(\omega, s) \in \mathcal{R} \times \{\varnothing, s_j\}$. Given a valuation v_j, this preference relation is represented by a utility function u_{s_j} that maps baskets (ω, s) into real numbers, we assume the following quasi-linear utility function for each seller s_j

$$u_{s_j}(\omega, s; v_j) = \begin{cases} \omega + v_j & \text{if } s = s_j, \\ \omega & \text{if } s = \varnothing. \end{cases}$$

Consider B, the set of n buyers. We identify a generic buyer by i. Each buyer i initially owns an amount of money $\omega_i \geq 0$, and no indivisible good. Thus, the initial endowment of buyer i is the basket $(\omega_i, \varnothing)$. Also, each buyer i has a valuation v_{ji} of good s_j, for all $s_j \in S$. So, the type of buyer i is a vector $\hat{v}_i = (v_{1i}, \ldots, v_{mi}, \omega_i) \in \mathbb{R}^m \times \mathbb{R}_+$. We denote by $\hat{V}_i$ the set of all possible types of buyer i. Also, each buyer i has a preference relation over baskets $(\omega, s) \in \mathbb{R} \times (S \cup \{\varnothing\})$. Given a type $\hat{v}_i$, this preference relation is represented by the utility function $u_i(\cdot)$ that maps baskets (ω, s) into real numbers. We assume the following quasi-linear utility function

$$u_i(\omega, s; \hat{v}_i) = \begin{cases} \omega + v_{ji} & \text{if } s = s_j, \\ \omega & \text{if } s = \varnothing. \end{cases}$$

The **state of the market** is the vector for all agents types $v = (v_1, \ldots, v_m, \hat{v}_1, \ldots, \hat{v}_n) \in \prod_j^m V_j \times \prod_{i=1}^n \hat{V}_i$. Let V be the set of all possible states of the market, i.e. $V = \prod_j^m V_j \times \prod_{i=1}^n \hat{V}_i$. We assume that the state of the market v is drawn according to a probability function f from V to $\mathbb{R}$, of common knowledge.

An **assignment** is a function Γ from $S \cup B$ to $\mathbb{R} \times (S \cup \{\varnothing\})$. We use $\Gamma(r) = (\Gamma_\omega(r), \Gamma_s(r))$ to describe the allocation of r under the assignment Γ, for all $r \in S \cup B$. That is to say, Γ assigns to each member of the market r a basket composed of an amount of money, $\Gamma_\omega(r)$, and at most one good, $\Gamma_s(r)$.

An assignment Γ is **feasible** if it satisfies the following three conditions:

1. $\sum_{r \in S \cup B} \Gamma_\omega(r) \leq \sum_{i=1}^{n} \omega_i$,

2. Let r, $r' \in S \cup B$. If $\Gamma_s(r) = \Gamma_s(r') \in S$, then $r = r'$, and

3. For all $s \in S$ there exists some $r \in S \cup B$ such that $\Gamma_s(r) = s$.

Conditions 2 and 3 tell us that at Γ, any good in the market is assigned to one and only one agent.

We say that a basket (ω, s) is **individually rational (IR)** for agent r if and only if the utility of r by holding (ω, s) is greater or equal than the utility of r by holding her initial endowment. So, Γ is an **individually rational (IR)** assignment if each member of the market weakly prefers her allocation under Γ to her initial endowment.

2.2 The Game

Agents interact in a two-stage game. Nature moves first determining the type of each member of the market according to the probability distribution f. All members of the market observe their type but do not observe the type of the others.

At stage 2, sellers decide simultaneously to set the price of their good. If a seller s_j decides to sell her good, she sets a non-negative price p_j. Otherwise, she sets a price $p_j = +\infty$. Thus, $A_j = \mathbb{R}_+ \cup \{+\infty\}$ is the set of actions of seller s_j. Consequently, a **price vector** $p = (p_1, p_2, \ldots, p_m)$ is an element of $A_1 \times A_2 \times \cdots A_m$.

At the end of the game, each sellers sells her good to a buyer for whom it is the preferred good among all goods, and he does not sell his good when his good is not the best good for some buyer. Payoffs are determined by the final assignment Λ, which is generated by a random procedure. Note that a seller s gets the basket $(p_s, \varnothing) = \Lambda(s)$ if and only if a buyer i gets the basket $(\omega_i - p_s, s) = \Lambda(i)$ and $u_i(\omega_i - p_s, s)$ is maximal.

2.3 The solution concept

To present the solution concept of our game, we introduce the following notation. A decision rule for seller s_j is a function $\sigma_j : V_j \to A_j$ mapping a type into a price. Thus, a **pure strategy** for seller s_j is an element $\sigma_j \in \Sigma_j = \{\sigma_j : \sigma_j \text{ is a decision rule}\}$. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ be the profile of sellers decision rules, where σ_{-s} denotes the profile of decision rules different to σ_s.

Sellers pay-offs depend on the final allocation Λ, which also depends on other sellers' strategies. Hence, each seller is assigned to a basket $\Lambda(s_j) = \Lambda(\sigma_j, \sigma_{-j})(s_j)$. Thus, a seller s sets her price by analyzing her expected utility function, denoted by $E[u_s]$. We present the concept solution in the following definition.

Definition 1. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_m^*)$ be a profile of pure strategies of sellers and buyers, respectively. The vector (σ^*) is a **Bayesian Nash equilibrium** if

$$E[u_{s_j}(\Lambda(\sigma_j^*, \sigma_{-j}^*)(s_j))] \geq E[u_{s_j}(\Lambda(\sigma_j, \sigma_{-j}^*)(s_j))],$$

for all $s_j \in S$ and $\sigma_j \in \Sigma_j$.

3 Equilibrium Analysis

To analyze the equilibrium of stage 2, we know that sellers simultaneously set their prices rules Given that each seller only knows her valuation, the solution of stage two is a profile of decision $(p_s(v_s))_{s \in S}$ such that each seller maximizes her expected utility function. That is to say, the concept solution for this stage is the Bayesian Nash equilibrium. Moreover, we know that the final expected utility of each seller depends on the assignment Λ that randomly assigns each sold good s_j to a buyer i for whom $v_{ji} - p_j$ is maximal. Note that the assignment procedure may generate many final allocations because the assignment procedure randomly assigns a good to some buyer who asks for it. Then, the set of sellers that sell their good does not change in all the possible assignments, and the only difference between them is the name of the buyers who buy a good. Therefore, sellers receive the same payoff regardless the final assignment.

The previous discussion implies that seller s_j earns p_j when she sells her good to some buyer i. Hence, the utility of s_j is independent of the name of the buyer dealing with only one of them. Consequently, the payoff function of s_j is given by

$$u_{s_j}(\omega, s; v_j) = \begin{cases} p_j & \text{if } s_j \text{ sells her good,} \\ v_j & \text{otherwise.} \end{cases}$$

To simplify the algebra, we consider $\overline{u}_{s_j} = u_{s_j} - v_j$ which is a monotonic transformation of the payoff function u_{s_j}. From now on, we use the following payoff function

$$\overline{u}_{s_j}(\omega, s; v_j) = \begin{cases} p_j - v_j & \text{if } s_j \text{ sells her good,} \\ 0 & \text{otherwise.} \end{cases}$$

3.1 Expected utility of sellers

An equilibrium decision rule for seller s_j is a decision rule σ_j that maximizes her expected utility function $E[u_{s_j}]$, which is equivalent to maximizing $E[\overline{u}_{s_j}]$. Formally, a decision rule, of seller s_j, maps a valuation v_j into a price p_j, i.e. $\sigma_j(v_j) = p_j$. Since seller s_j does not know buyers' valuations, s_j may gain $p_j - v_j$, in case of selling her good, or 0, otherwise. Given these two possible cases, the expected utility of s_j is

$$E[\overline{u}_{s_j}] = (p_j - v_j)Pr[Selling] + 0Pr[\text{Not selling}].$$

Let $z_i^j = v_{ji} - p_j$ be the surplus of i when she buys good s_j. Note that, if some good s_j is bought to some buyer i, the assignment procedure and the fact that buyers are not strategic guarantee that the basket $(\omega_j - p_j, s_j)$ is the top basket of i. Since we assume a quasi-linear utility function, buyers buy a good that provides the largest positive surplus since we assume a quasi-linear utility function. Thus, the probability that s_j sells her good can be described as follows

$$Pr[Selling] = Pr\left[z_i^j \geq \max_{s_\tau \in S}\left\{z_\tau^i \mid z_\tau^i \geq 0 \text{ and } \tau \neq j\right\} \text{ for some } i \in B\right].$$

Consequently, we can re-write the expected utility of s_j as follows

$$E[\overline{u}_{s_j}] = (p_j - v_j)Pr\left[z_i^j \geq \max_{s_\tau \in S}\left\{z_\tau^i \mid z_\tau^i \geq 0 \text{ and } \tau \neq j\right\} \text{ for some } i \in B\right]. \tag{1}$$

3.2 Sellers best responses

To determine the best responses at equilibrium we proceed by maximizing the expected utility $E[\overline{u}_{s_j}]$ of each seller s_j. Then, it is necessary to compute the probability $Pr[Selling]$, that corresponds to the event where the surplus z_i^j is maximal for some buyer i. This surplus depends of buyers i valuation vector $\hat{v}_i$, and the price vector p, set at the end of stage two. We know that sellers do not observe buyers valuations vector and other sellers prices, then $\hat{v}_i$ is the realization of the random vector $\hat{V}_i = (V_{1i}, V_{2i}, \ldots, V_{mi})$, and p_τ is the realization of the random variable $P_\tau = \sigma_\tau(V_j)$. However, under our Bayesian framework, we have that the probability distribution f of the random vector $V = (V_1, V_2, \ldots, V_m, \hat{V}_1, \ldots, \hat{V}_n)$ is common knowledge. Moreover, we assume that random variables V_τ and $V_{\tau i}$ are independent and identically distributed for all $\tau \neq i$ and $i = 1, 2, \ldots, n$.

Given the description of $Pr[Selling]$, we use $V_M := \max_{s_\tau \in S}\{z^i_\tau | z^i_\tau \geq 0 \text{ and } \tau \neq j\}$. In this setting, it is important to remark that sellers do not necessarily follow a symmetric behavior because V_M is the largest surplus, or the maximum order statistic. Also, V_m is a non-negative random variable because each surplus z^i_s is a random variable, for all $s \in S$, and each buyer only asks for a good if it provides her a positive payoff. This notation allows us to rewrite the expected utility function (1) in the following way

$$E[\overline{u}_{s_j}] \;=\; (p_j - v_j)Pr[v_{ji} - p_j > v_M \text{ and } v_M \geq 0]. \tag{2}$$

Expression 2 establishes the way to compute $Pr[Selling]$. In words, we get this probability by integrating the joint distribution of V_{ji} and V_M, namely $f_{V_{ji}V_M}$, over the probability region $R = \{(v_M, v_{ji}) \in \mathbb{R}^2 \mid v_{ji} - p_j > v_M \text{ and } v_M \geq 0\}$. Graphically, we see this region of integration in Figure 1.

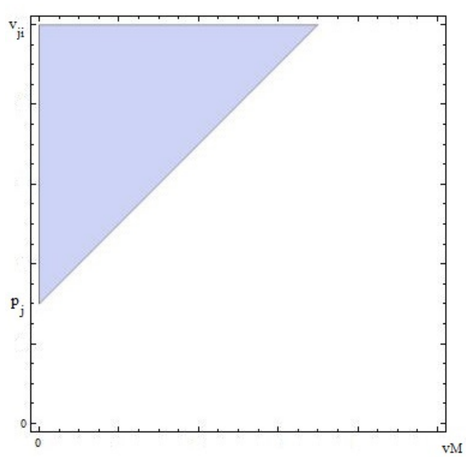

Figure 1: Probability region.

Remember that we assume that V_j and $\hat{V}_i$ are statistically independent for all $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Hence, V_{ji} and V_M are statistically independent because V_M is the largest surplus among all sellers surpluses, excluding the surplus Z^i_j; V_M is a transformation of random variables that are statistically with V_j. Consequently, the joint distribution between V_{ji} and V_M is equal to the product of their marginal distributions. So, we have that $f_{V_{ji}V_M} = f_{V_{ji}} f_{V_M}$, and

$$\begin{aligned} Pr[Selling] &= \int_{p_j}^{\infty} \int_0^{v_{ji}-p_j} f_{V_{ji}}(v_{ji}) f_{V_M}(v_M) dv_M dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(\int_0^{v_{ji}-p_j} f_{V_M}(v_M) dv_M \right) dv_{ji} \\ &= \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) \left(F_{V_M}(v_{ji}-p_j) - F_{V_M}(0) \right) dv_{ji}, \end{aligned}$$

where F_{V_M} is the cumulative distribution function of V_M, a non-negative variable because buyers do not ask for unacceptable goods. This means that $F_{V_M}(0) = 0$, and

$$Pr[Selling] = \int_{p_j}^{\infty} f_{V_{ji}}(v_{ji}) F_{V_M}(v_{ji} - p_j) dv_{ji}. \tag{3}$$

Now, consider the change of variable $x = v_{ji} - p_j$ that implies $dx = dv_{ji}$ and $v_{ji} = x + p_j$. We substitute v_{ji} by x into expression (3). We get the following integral

$$Pr[Selling] = \int_0^{\infty} f_{V_{ji}}(x + p_j) F_{V_M}(x) dx. \tag{4}$$

We know that F_{V_M} is right continuous, which implies its integrability on any interval $[a, b]$, with $a \geq 0$. Also, this cumulative distribution is upper and lower bounded by 1 and 0, respectively. Since the distribution function $f_{V_{ji}}$ is non-negative, we can apply the First Mean Value Theorem for infinite integrals [Gradsteyn and Ryzhik, 2014] on expression (4). Then, there exits $\mu \in [0, 1]$ such that

$$\begin{aligned} Pr[Selling] &= \mu \int_0^{\infty} f_{V_{ji}}(x + p_j) dx \\ &= \mu(1 - F_{V_{ji}}(p_j)). \end{aligned}$$

The constant μ is the mean of the function f_{V_M} on the interval $[0, v_{ji} - p_j]$. This implies that μ is the mean of the maximum surplus V_M, excluding $v_{ji} - p_j$. Hence, $Pr[Selling]$ tells us that s_j only cares to provide i a greater surplus than the mean of V_M, considering that other sellers profile of prices is $p_{-s_j} = (p_\tau(V_\tau))_{\tau \neq j}$.

By the previous discussion, the expected utility function of seller s_j is

$$E[\overline{u}_{s_j}] = \mu(p_j - v_j) \left(1 - F_{V_{ji}}(p_j)\right). \tag{5}$$

To maximize the expected utility of seller s_j, we have to solve the first order condition, i.e. $\partial E[\overline{u}s_j]/\partial p_j = 0$. Hence, we derive expression (5) with respect to p_j, which implies that seller s_j best response is the solution of the equation

$$\mu\left(1 - F_{V_{ji}}(p_j)\right) - \mu(p_j - v_j)f_{v_{ji}}(p_j) = 0.$$

Rearranging the previous expression, we get that

$$\left(1 - F_{V_{ji}}(p_j)\right) = (p_j - v_j)f_{V_{ji}}(p_j).$$

This means that s_j best responses balance the direct effect of increasing the price with the indirect effect of not selling the good. Even more, we can rearrange the previous expression in the following way

$$p_j = \frac{1 - F_{V_{ji}}(p_j)}{f_{Vji}(p_j)} + v_j. \tag{6}$$

Expression (6) implies that best decision rules $p_j^*(v_j)$ of s_j are implicitly defined by an equation which depends on the inverse hazard rate of buyer i. Considering

$$\gamma_j(p_j) = \frac{1 - F_{V_{ji}}(p_j)}{f_{Vji}(p_j)} + v_j, \tag{7}$$

we have that best responses satisfy the following property

$$\gamma_j(p_j) = p_j,$$

where γ can be a linear or non-linear function. Therefore, best responses of seller s_j are fixed points of (7).

3.3 Existence and uniqueness conditions

As we noted in the previous section, sellers' best responses are fixed points of the function γ_j (see expression (7)) which is not necessarily a linear function. To find fixed points of non-linear functions, it is common the use of numerical techniques like the Newton's Method. In this section, we part from this method to analyze the existence and uniqueness of sellers decisions rules in our Bayesian version of the assignment game. Below, we describe Newton's Method to compute the solutions of a non-linear equation, to later explain its relation with fixed points.

Consider an equation $g(x) = 0$, where g is a non-linear function. Suppose that this equation has at least one **root** $x^* \in \mathbb{R}$, i.e. $g(x^*) = 0$. The Newton's Method proceeds as follows

Step 0. Start with an initial guess, $x_0 \in \mathbb{R}$, for the location of the root.

Step t. To find an equation's root, we improve the initial guessing by iterating repeatedly the next expression

$$x_{t+1} = x_t - \frac{g(x_t)}{g'(x_t)}.$$

Previous procedure generates the set $\{x_t\}_{t=0}^{\infty} = \{x_0, x_1, \ldots x_t, \ldots\}$, which is called the Newton's succession. It is possible to demonstrate that $x^* = \lim_{t\to\infty} x_t$ is a root of the non-linear equation $g(x) = 0$ [Palais, 2007].

Two immediate questions arise about Newton's method application. The first one is related to the convergence of Newton's succession. The second one is about the independence of the initial guess, i.e., Does different initial guesses get the same convergence point? To answer these questions, it is important to note that, if Newton's succession $\{x_t\}_{t\in\mathbb{N}}$ converges to some x^*, we have that

$$x^* = x^* - \frac{g(x^*)}{g'(x^*)}.$$

In other words, the point x^* is a **fixed point** of the function

$$h(x) = x - \frac{g(x)}{g'(x)}.$$

This means that finding a unique root for equation $g(x) = 0$ is equivalent to find a unique fixed point of function h. To analyze the uniqueness of fixed points, it is necessary to introduce the definition of contractions.

Definition 2. A function $g : \mathbb{R} \to \mathbb{R}$ is a **contraction** if there exists a constant L such that $0 < L < 1$ for any $x, y \in \mathbb{R}$:

$$|g(x) - g(y)| \leq L|x - y|.$$

The contraction property is a sufficient condition to guarantee the convergence of Newton's succession. Even more, this property makes the convergence process independent of the initial guess as it is stated in the following theorem.

Theorem 3.1. ***(Contracting Map, Banach [1922])****. Consider $h : X \subset \mathbb{R} \to X$ a contraction. The function h has a unique fixed point $x^* \in X$. Also, the Newton's succession converges to x^* as $n \to \infty$ for any $x_0 \in X$.*

Proof. A sketch of the proof is shown in the Appendix. For more details, see [Palais, 2007]. □

The Contracting Map Theorem guarantees the existence of a unique fixed point when functions are contractions. Even more, Newton's Method establishes a link between finding fixed points, and the roots of a non-linear equation. Hence, if function (7) is a contraction, the Theorem 3.1 implies that sellers best response is unique.

Theorem 3.2. *Consider a market where buyers are non-strategic, goods are assigned to a buyer who considers it her best good, and*

$$\gamma_j(p_j) = \frac{1 - F_{V_{ji}(p_j)}}{f_{V_{ji}}(p_j)} + v_j.$$

If γ_j is a contraction map on a subset X of $\mathbb{R}$, then γ_j has a unique fixed point p_j^*. Even more, the Newton's succession

$$p_{n+1} = p_n - \frac{\gamma_j(p_n) - p_n}{\gamma_j'(p_n) - 1}$$

converges to p_j^* regardless the initial guess $p_0 \in X$.

Proof. It is a consequence of Theorem 3.1. □

Theorem 3.2 establishes that s_j best response is unique when γ_j is a contraction. Even more, we can use the Newton's Method to find it.

Remark 1. Geometrical Interpretation of the Contraction Property. If a function g is a contraction, this means that the function reduces the distance between their images. Moreover, we can rewrite this condition as follows

$$\frac{|g(x) - g(y)|}{|x - y|} \leq L, \text{ for some } 0 \leq L < 1.$$

Hence, the slope of the tangent line of $g(x)$ is bounded by L, which makes them very similar to a constant or linear functions.

In our case, we impose the contraction property over $\gamma = (1 - F_{V_{ji}})/f_{V_{ji}}$, to analyse the uniqueness of equilibrium prices, which means that the inverse hazard rate function is similar to a constant/linear function by Remark 1. Hence, we can say that $1 - F_{v_{ji}}$ is, approximately, multiple of $f_{V_{ji}}$.

The following example shows that multiple equilibria arise when the function γ is not a contraction.

Example 3.1. Multiple equilibrium. By Theorem 3.2, the bests responses of seller s_j depend on the distribution $f_{V_{ji}}$. We consider the following distribution function:

$$f_{V_{ji}}(v) = \begin{cases} \frac{1+v^4}{630} & \text{if } 0 \leq v \leq 5, \\ 0 & \text{otherwise.} \end{cases} .$$

Hence, the cumulative distribution function is

$$F_{V_{ji}}(v) = \begin{cases} 0 & \text{if } v \leq 0 \\ \frac{v}{630} + \frac{v^5}{3150} & \text{if } 0 \leq v \leq 5 \\ 0 & \text{if } v \geq 5. \end{cases} . \tag{8}$$

Considering $f_{V_{ji}}$ and $F_{V_{ji}}$, the equation that implicitly defines best responses of seller s_j is

$$\frac{p_j^5}{525} - \frac{v_j p_j^4}{630} + \frac{p_j}{315} - \frac{v_j}{630} - 1 = 0.$$

Then, bests responses are solutions of a five degree equation, which at most have five roots. According to the Descartes' Rule, it is possible to approximate the number of positive roots by counting the number of sign changes. In the previous equation, we have three changes of sign, which means that seller s_j has at most three positive real roots, and at least one real root. We analyse the multiplicity of positive roots through the contraction condition. By doing some algebra, we get that

$$\gamma(p_j) = \frac{1 - F_{V_{ji}}(p_j)}{f_{v_{ji}}(p_j)} = \frac{3150 - 5p_j - p_j^5}{5(1 + p_j^4)} \text{ for all } p_j \in [0, 5].$$

Now note that

$$|\gamma(0) - \gamma(1)| = \frac{1578}{5} > 1,$$

this means that γ is not a contraction on the interval $[0, 5]$, where it is defined. Hence, the probability distribution (8) does not induce a unique best response for seller s_j. Therefore, equilibrium prices are not unique.

□

Despite that multiple equilibria arise when γ is not a contraction, in the following section we show that this condition is not necessary. In other words, it is sufficient to check if γ satisfies such condition to verify the uniqueness of equilibrium prices, but it is possible to have a unique equilibrium even if γ is not a contraction.

Although the equilibrium price is implicitly defined by expression (7), assuming that such function is a contraction allows us to analyse the relationship between p_j^* and v_j. This relationship is positive when the density function of V_{ji} is positive.

Proposition 3.1. *The relationship between p_j^* and v_j is positive when $\partial f_{V_{ji}}(p_j)/\partial p_j \geq 0$.*

Proof. We know that p_j^* is implicitly defined by the equation

$$(1 - F_{V_{ji}}(p_j^*)) = (p_j^* - v_j) f_{V_{ji}}(p_j^*).$$

The implicit derivative with respect to v_j is

$$\begin{aligned} -f_{V_{ji}}(p_j^*)\frac{dp_j^*}{dv_j} &= \left(\frac{dp_j^*}{v_j} - 1\right) f_{V_{ji}}(p_j^*) + (p_j^* - v_j)\frac{df_{V_{ji}}(p_j^*)}{dp_j^*}\frac{dp_j^*}{dv_j} \\ f_{V_{ji}}(p_j^*) &= \frac{dp_j^*}{dv_j}\left(2f_{V_{ji}}(p_j) + (p_j^* - v_j)\frac{df_{V_{ji}}(p_j^*)}{dp_j^*}\right) \end{aligned}$$

In case of selling, we know that $(p_j^* - v_j) > 0$. When $\partial f_{V_{ji}}(p_j)/\partial p_j \geq 0$, we have that

$$2f_{V_{ji}}(p_j) + (p_j^* - v_j)\frac{df_{V_{ji}}(p_j^*)}{dp_j^*} > 0$$

because $f_{V_{ji}}$ is a density function. We conclude that

$$\frac{dp_j^*}{dv_j} > 0.$$

□

3.4 The uniform distribution

Remark 1 incentives us to verify if the uniform distribution, a constant distribution, induces a unique price vector at equilibrium. Assuming that V_{ji} is uniformly distributed on an interval $[a, b]$, we have that

$$f_{V_{ji}}(v) = \begin{cases} \frac{1}{b-a} & \text{for all } v \in [a,b], \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } F_{V_{ji}}(v) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } x \in [a,b] \\ 1 & \text{if } x \geq b. \end{cases}$$

Substituting in expression (6), we get that

$$\begin{aligned} p_j &= \frac{1 - \frac{b_j - a}{b-a}}{\frac{1}{b-a}} + v_j \\ &= b - p_j + v. \end{aligned}$$

Therefore, the equilibrium price is $p_j^* = b/2 + v_j$. However, considering γ as it is defined in expression (7), we note the following

$$|\gamma(x) - \gamma(y)| = |b - x - (b - y)| = |x - y|.$$

Hence, γ is not a contraction when buyers' valuations are uniformly distributed. This means that the contraction requirement over γ is sufficient, but not necessary.

3.5 Particular case: low price auctions

In this section, we discuss how sellers equilibrium behavior in our Bayesian setting is similar to an auction where a seller sells her good by offering the lowest price to the buyers who are willing to pay the price. To simplify the explanation, we consider that all goods are homogeneous, which means that buyers have the same valuation for all of them, i.e. $v_{ji} = v_{ki}$ for all $s_j, s_k \in S$. Also, we assume that buyers valuations are common knowledge. So, buyer i buys good s_j if and only if $v_{ji} - p_j > v_{ki} - p_k$ for all $s_k \in S - \{s_j\}$, i.e. seller s_j sells her good by setting a price who provides i the largest surplus. Since we assume that i is indifferent between all the goods in the market, s_j sells her good to i if and only if she sets the lowest price.

At equilibrium, we know that p_j^* is seller s_j best response of other sellers best responses, denoted by $p_k^* = \sigma^*(v_k)$ for all $k \neq j$. When i buys the good s_j, the discussion in the previous paragraph guarantees that $p_j^* < \min_{s_k \neq s_j}\{p_k^* = \sigma^*(v_k)\}$. Assuming that $\sigma^* = (\sigma_k^*)_{k\neq j}$ is a symmetric profile of increasing best responses, we have that

$$\min_{s_k \neq s_j}\{p_k^* = \sigma^*\} = \sigma^*(V_m = \min_{s_k \neq s_j}\{v_k\}).$$

The fact that σ^* is increasing also implies that its inverse function $(\sigma^*)^{-1}$ exists. Thus, s_j sells her good to i if and only if $(\sigma^*)^{-1}(p_j^*) < V_m$. Consequently, $Pr[Selling] = Pr[(\sigma^*)^{-1}(p_j^*) < V_m]$. Substituting the previous discussion in the expected utility of s_j, we get that

$$E[\overline{u}_{s_j}] = (v_j - p_j)Pr[(\sigma^*)^{-1}(p_j^*) < V_m].$$

Let F_{V_m} be the cumulative probability function of V_m, then

$$E[\overline{u}_{s_j}] = (v_j - p_j)(1 - F_{V_m}(\sigma^*)^{-1}(p_j^*)). \tag{9}$$

When buyers valuations are private information and goods are non-homogeneous, we cannot ensure that s_j sets the lowest price for buyer i. However, note that expected utilities in both cases, functions (5) and (9) respectively, have the same structure because a good is sold when a buyer maximizes her surplus with it. Consequently, in the general case we can conclude that each seller focuses on setting a price for which some buyer is willing to pay.

Now, we investigate the features of the equilibrium price. To find the best responses at equilibrium, we compute the first derivative of $E[\overline{u}_{s_j}]$ by applying the Inverse Function Theorem. Then, s_j best responses are the solutions of the following equation

$$-(1 - F_{V_m}((\sigma^*)^{-1}(p_j^*)) + (v_j - p_j)\left(-\frac{f_{V_m}((\sigma^*)^{-1}(p_j))}{(\sigma^*)'((\sigma^*)^{-1}(p_j))}\right) \quad = \quad 0.$$

Let $p_j^* = \sigma^*(v_i)$ be the best response of s_j at a unique symmetric equilibrium. This best response satisfies the first order condition, i.e., which means that

$$\begin{aligned}
-(1 - F_{V_m}(v_j)) + (v_j - \sigma^*(p_j))\frac{-f_{V_m}(v_j)}{(\sigma^*)'(v_j)} &= 0 \\
-(\sigma^*)'(v_j)(1 - F_{V_m}(v_j)) + (v_j - \sigma^*(p_j))(-f_{V_m}(v_j)) &= 0 \\
-(\sigma^*)'(v_j)(1 - F_{V_m}(v_j)) - \sigma^*(p_j)(-f_{V_m}(v_j)) &= v_j f_{V_m}(v_j)
\end{aligned}$$

Note that

$$\frac{d[-\sigma^*(v_j)(1 - F_{V_m}(v_j))]}{dv_j} = -(\sigma^*)'(v_j)(1 - F_{V_m}(v_j)) - \sigma^*(p_j)(-f_{V_m}(v_j),$$

then

$$\frac{d[-\sigma^*(v_j)(1 - F_{V_m}(v_j))]}{dv_j} \quad = \quad v_j f_{V_m}(v_j). \tag{10}$$

Integrating expression (10), we conclude that

$$\sigma^*(v_j) = \frac{-1}{1 - F_{V_m}(v_m)} \int_0^{v_j} \tau f_{V_m}(\tau) d\tau. \tag{11}$$

Expression (11) means that the best response of s_j is the expectation of V_m conditional to those values greater than v_j, i.e.

$$\sigma^*(v_j) = E[V_m | V_m > v_j],$$

which means that s_j sells her good when she expects than other sellers valuations are greater than her valuation. Hence, restricting the analysis to homogeneous goods and assuming that buyers valuations are common knowledge, we observe that equilibrium prices are explicitly defined, which is not the case when goods are heterogeneous and buyers valuations are private information (see expression (7)). However, in both cases, it is clear that p_j^* depends on v_j, as the following proposition illustrates. Even more, the relationship between them is positive, as in Proposition 3.1.

Proposition 3.2. *At equilibrium, s_j sets prices above her valuation.*

Proof. Integrating by parts expression (11), we get that

$$\begin{aligned}
\sigma^*(v_j) &= \frac{-1}{1-F_{V_m}(v_j)}\left(\tau\, F_{V_j}(\tau)\Big|_0^{v_j} - \int_0^{v_j} F_{V_m}(\tau)d\tau\right)\\
&= \frac{-1}{1-F_{V_m}(v_j)}\left(v_j F_{V_j}(v_j) + v_j - v_j - \int_0^{v_j} F_{V_m}(\tau)d\tau\right)\\
&= \frac{-1}{1-F_{V_m}(v_j)}\left(v_j F_{V_m}(v_j) + v_j - v_j - \int_0^{v_j} F_{V_m}(\tau)d\tau\right)\\
&= \frac{1}{1-F_{V_m}(v_j)}\left(v_j(1-F_{V_m}(v_j)) + \int_0^{v_j} F_{V_m}(\tau)d\tau - \int_0^{v_j} 1d\tau\right)\\
&= v_j - \frac{1}{1-F_{V_m}(v_j)}\int_{v_j}^{0}(F_{V_m}(\tau)-1)d\tau.
\end{aligned}$$

Note that $F_{V_m}(v_j) - 1 < 0$, hence $-\int_{v_j}^0 (F_{V_m}(\tau)-1)d\tau > 0$.

□

4 Equilibrium Characterization for the Exponential Case

In this section, we show that an exponential distribution induces a function γ that satisfies the contraction condition, and consequently behaves like in explanation (1). Even more, this distribution function allows us to compute a closed form solution to the Bayesian Nash equilibrium, which is suitable to perform some comparative statistics for different probability distribution assumptions.

4.1 Identically distributed random variables

Consider that V_j, V_{ji} are independent and exponentially distributed with parameter $\lambda > 0$. So, their probability distributions are

$$f_{V_{ji}}(x) = f_{V_j}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0,\\ 0 & \text{otherwise,}\end{cases}$$

for all $j \in \{1, 2, \ldots, m\}$ and $i \in \{1, 2, \ldots, n\}$. The mean of this distribution, $1/\lambda$, is the occurrence of selling the indivisible good. Thus, if λ increases, buying the object happens more quickly. By expression (5), we have that

$$E[\overline{u}_{s_j}] = (p_j - v_j)\mu e^{-\lambda p_j}. \tag{12}$$

Although it is not difficult to solve the first order condition of $E[\overline{u}_{s_j}]$ when V_{ji} follows an exponential distribution, in the following proposition we show that $\gamma = (1 - F_{V_{ji}})/f_{V_{ji}}$ is a contraction map to show the application of Theorem 3.2.

Proposition 4.1. *The function γ is a contraction map when V_{ji} is exponentially distributed.*

Proof. We have that

$$\begin{aligned} \gamma &= \frac{1 - F_{V_{ji}}}{f_{V_{ji}}} \\ &= \frac{1 - (1 - e^{-\lambda x})}{\lambda e^{-\lambda}} \\ &= \frac{1}{\lambda}. \end{aligned}$$

Hence, γ is a constant function when V_{ji} is exponentially distributed. Clearly, γ is a contraction. □

To derive a closed for for the equilibrium price, we use expression (6). Consequently, we get that

$$p_j = \frac{1}{\lambda} + v_j.$$

The following theorem summarizes the previous discussion.

Theorem 4.1. *Suppose that V_j and V_{ji} are independent and exponentially distributed with parameter $\lambda > 0$. The price that each seller s_j sets at equilibrium is*

$$p_j^*(v_j) = \frac{1}{\lambda} + v_j,$$

for all $s_j \in S$.

Since the decision rule at equilibrium is unique, we can do some comparative statics.

Corollary 4.1. *Let p_j^* be the unique price at the symmetric equilibrium found. Then*

1. *The relation between* p_j^* *and* v_j *is positive, and*

2. *the relation between* p_j^* *and* λ *is negative.*

Proof. By Theorem 4.1, we know that $p_j^* = 1/(m+1)\lambda + v_j$. Taking the derivatives of p_j^* with respect to v_j, λ and m, we get that

$$\begin{aligned} \frac{\partial p_j^*}{\partial v_j} &= 1 > 0, \\ \frac{\partial p_j^*}{\partial \lambda} &= -\frac{1}{\lambda^2} < 0. \end{aligned}$$

□

In other words, the price increases when the valuation of seller s_j increases, and decreases when the parameter λ increases. This last point implies that prices increase when buyers are in a hurry to buy an indivisible good.

4.2 Non-identically distributed random variables

By expression (7), we know that equilibrium characterization only depends on how the buyer i values the good. Thus, uniqueness of selling prices do not change when we assume that variables $V_{j\tau}$ and V_j are not identically distributed for all $\tau \in B$ and $s_j \in S$. For example, if we consider that each random variable V_{ji} is exponentially distributed with parameter λ_i, it is easy to see that equilibrium price vector is

$$p_j^* = \frac{1}{\lambda_i} + v_j.$$

4.3 The number of sellers in the market

In the case of valuations that are exponentially distributed, the interpretation of the distribution's parameter is interesting. As we mentioned before, its inverse is the mean of the distribution, which means how fast the event, buying an object, happens. So, when $1/\lambda$ tends to zero, the buyer wants to buy a house more quickly. This observation allows us to relate λ to the number of sellers in the market.

Head, Lloyd-Ellis, and Sun [2014] find that buyers delay buying a house when there is a large number of sellers. Regarding the exponential distribution, this means that the mean of the buying event increases as the number of sellers increases. So, we have that $1/\lambda$ depends on the number of sellers, i.e. $1/\lambda = 1/\lambda(m)$, and there is a positive relationship between $1/\lambda$ and m. For example, we can assume that $1/\lambda(m) = m$, i.e., the buyer checks all objects, one at a time, to buy one. In this case, the Theorem 4.1 implies that

$$p_j^* = \frac{1}{\lambda}(m) + v_j.$$

Therefore

$$\frac{\partial p_j^*}{\partial m} = \frac{d(1/\lambda)}{dm} + 0 > 0.$$

In words, equilibrium prices increase when the number of sellers increases.

4.4 Overlapping Valuations

In this section, we analyze the situation where valuations are overlapped but exponentially distributed. So, we assume that variables V_j and V_{ji} have a minimum value r_j and r_{ji}, respectively. Consequently, their probability distributions are

$$f_{V_j}(v_j) = \begin{cases} \lambda_j e^{-\lambda_j(v_j - r_j)} & \text{if } v_j > r_j, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad f_{V_{ji}}(v_{ji}) = \begin{cases} \lambda_{ji} e^{-\lambda_{ji}(v_{ji} - r_{ji})} & \text{if } v_{ji} > r_{ji}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently we have that $F_{V_{ji}}(p_j) = 1 - e^{-\lambda_{ji}(p_j - r_{ji})}$. By substituting in expression (6), we get that

$$p_j^* = \frac{e^{-\lambda_{ji}(p_j - r_{ji})}}{\lambda_{ji} e^{-\lambda_{ji}(p_j - r_{ji})}} = \frac{1}{\lambda_{ji}} + v_j.$$

Therefore, comparative statics are similar to the ones presented in Corollary 4.1.

5 Concluding Remarks

We analyze the uniqueness of equilibrium prices in a Bayesian version of the Assignment Game assuming that sellers are strategic, but buyers not. In this framework, we show that equilibrium prices are determined by the features of the valuation distribution function of the buyer who gets good. Even more, we find that quotient between the cumulative distribution function and the density function that are contraction induce the existence of unique price equilibrium.

The Contracting Map theorem indicates us that buyers valuations distribution is similar to constant/linear function. So, we demonstrate the exponential probability functions satisfy the contraction's requirements, and we get a closed form solution via this type of distributions. Under the exponential distribution, the uniqueness prevails even if valuations are not identical, or if they are overlapped. Also, comparative statistics are naturally performed in this closed form solution, reflecting empirical evidence facts.

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A Contracting Map Theorem

Theorem A.1. *(**Contraction Mapping**) Assume that $g(x)$ is a continuous function on $[a, b]$. Also, suppose that $g(x)$ satisfies the Lipschitz condition (2), and that $g([a, b]) \subseteq [a, b]$. Then $g(x)$ has a unique fixed point $c \in [a, b]$. Also, the Newton's succession $\{x_n\}$ defined in the main text converges to c as $n \to \infty$ for any $x_0 \in [a, b]$.*

Proof. By the Brower's Theorem, we know that $g(x)$ has at least one fixed point. So, to prove the uniqueness of the fixed point, we assume that there are two fixed points c_1 and c_2. We will prove that these two points must be identical. We know that

$$|c_1 - c_2| = |g(c_1) - g(c_2)| \leq L|c_1 - c_2| \text{ and } 0 < L < 1,$$

consequently, c_1 must be equal to c_2.

Finally, we need to prove that the succession described in the main text converge to c, for any $x_0 \in [a, b]$. note that

$$|x_{n+1} - c| = |g(x_n) - g(c)| \leq L|x_n - c| \leq \ldots \leq L^{n+1}|x_0 - c|.$$

Since $0 < L < 1$, we have that $|x_{n+1} - c| \to 0$, as $n \to \infty$. The succession converges to the fixed point of $g(x)$, independently of the starting point x_0. □