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AUCTION THEORY. A GUIDED TOUR

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DOCUMENTO DE TRABAJO

Núm. II - 1998

Auction Theory;
A guided tour

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May, 1998

Abstract

I present an overview of what has been the theoretical analysis of auctions in the last two decades. The goal is to offer a systematic exposition of the main issues addressed by this literature with more emphasis on why than on what. For that, I present a unified framework that takes the reader through the analysis of both design questions and positive ones dealing with the workings of several standard institutions. At the end of the tour, the reader will find him or herself inside the realm of market design and the theory of price formation.

JEL Classification number: D44

1.- Introduction

Since the pioneer work by Vickrey in the early sixties, the economic analysis of auctions has known two waves of high activity. The first one, starting in the early eighties, had its roots in the invasion of economics by game theory. Soon, this old institution was the aim of game theoretic analysis. Not only game theory made it possible to undertake a systematic analysis of the strategic interaction in auctions, but also an auction constituted a neatly defined set of rules and outcomes, one that immediately rendered itself as a game. Not surprisingly, the first years of that decade witnessed the lay down of what is still the basis of auction theory.[1]

The second wave is one that is perhaps still in the ascending phase and was caused by what has been referred to as the success story in game theory: the US Federal Communications Commission auctions of spectrum rights. These auctions, which started in 1994, put in the hands of private firms rights on radio spectrum that will become one of the supports of the new personal communications systems. Major auction theorists were in charge of designing these complex auctions as a substitute for administrative processes considered slow, allocationally inefficient, and weak at raising revenue. The first evaluation of the performance of these auctions were very positive and portrayed auction theory as a useful tool to design the economic reality. These auctions, however, have shown how many important issues are yet unexplored in auction theory.

But, one could ask what is the real importance of the economic analysis of auctions in the wide territory of economic theory. A common answer to that question refers to the widespread use of one form of auctions or another for the trade of a good deal of commodities. Indeed, auctions are not only used for selling art or wine. Mineral rights and timber are usually allocated in this fashion. Public works contracts are typically awarded through competitive bidding as well. But so are treasury bills and other monetary instruments. Organized markets like stock exchanges can also be characterized as some type of auction. And recent reform in several electric generation markets, like that undertaken in the UK, Spain, or California, are in the direction of introducing competitive bidding as a way

[1] Surveys of this literature with an emphasis different from the one here can be found in McAfee and McMillan (1987b), Milgrom (1985), Milgrom (1989), or Wilson (1992), and a more recent one in Wolfstetter (1996).

of assigning generation to plants.

I would like to emphasize another reason to study auction theory: the grounds it offers on which to build at least one theory of both market design and price formation. This is undoubtedly central to economic theory and for the most part still missing.[2] Indeed, the paradigmatic model of a market, the Walrasian model, postulates a large number of sellers and buyers all of whom take prices as given, and then shows under what circumstances there is a price (vector) that clears the market. But there is no compelling story in that model as to who sets that price or how it becomes the market price. Auction theory takes, in a sense, the opposite approach. From the seminal work by Vickrey (1961), auction theory starts by defining the rules of trade, and then analyze the behavior of agents under those rules. There is no reference to what would be an "equilibrium price", but rather the terms of trade and the trade itself are determined by the explicit interaction of agents acting under the specified set of rules.

An auction, it may be argued, is not a market in several respects. First, in a market there is the potential for trade of a large number of homogeneous units of a good. Second, in a market there is no parallel to the "seller" or auctioneer, a peculiar side of a market. But both these objections have been overruled by the development of auction theory. Indeed, if the first works in the area analyzed the auction of a single, indivisible good, soon it became clear that the same approach could be used to analyze the sale of multiple units of the good. Also, there was no conceptual need for the strong asymmetry between one side of the "market" and the other. If more difficult, still the analysis of two sided bidding, where buyers and seller would both bid their maximum and minimum acceptable price, respectively, was feasible. Moreover, even decentralized competition, one in which sellers and buyers don't use the services of a "clearing house" or market maker, are apt for analysis using the tools and approach of auction theory.

Auction theory has not advance only in the positive analysis of trading institutions. In fact, most of the work in the area, at least in the eighties, has been of the normative type. Here too, Vickrey made the first contribution by defining what became known as the

[2] Of course, auction theory is not the only way to do it. Search and bargaining theory are alternatives, if not rival, to auction theory. See Osborne and Rubistein (1990) for a good survey of these literatures.

buyers in each of those auctions mentioned above.

Consider the second price, sealed bid auction. It is very easy to see that the best bid for a buyer i is exactly her valuation v_i. Indeed, for buyer i and for any bids submitted by the rivals, winning the auction, i.e. bidding above the highest rival bid, means paying that highest rival bid. And buyer i is interested in winning under those conditions if and only if the highest rival bid is below her own valuation, which is what she guarantees by bidding $B(v_i) = v_i$.

Now assume that the seller selects the first price method. Then bidding her own valuation is not a very clever strategy for buyer i. Indeed, doing so she gets zero surplus (rents, from now on) both when winning and when loosing the auction, whereas by submitting a bid lower than her valuation she may still win the auction securing some positive rents in that case. That is, a change in the trading rules induces a change in the traders' behavior. But, how much should buyer i reduce her bid? The answer to this question is going to be a little bit more involved. The reason is that, contrary to the second price case, now the best bid for buyer i is going to depend on what other buyers bid. That is, there is no dominant strategy in this game!

Thus, we should now look for a (Bayesian) equilibrium of the game, that is, a vector of bidding functions with the property that, any buyer i with any valuation will find it optimal to bid according to her respective bidding function if she expects everybody else to do likewise. It is natural to expect that all buyers use the same bidding function (not the same bid) and that this be strictly monotone. Indeed, the first price auction game treats buyers symmetrically and buyers with higher valuation should be willing to pay more to increase the probability of winning the object. Thus, let us postulate a (common) monotone equilibrium bidding function $b(v_i)$. From monotonicity, the winner will be the one with highest valuation and then her expected rents will be

$$[v_i - b(v_i)]F(v_i)^{N-1} \qquad (1)$$

But for $b(v_i)$ to actually be the best bid for buyer i, these rents should be higher than the rents she expect if bidding any other amount, for instance $b(z)$, for any arbitrary z, that is, by "pretending" to have any other valuation. Therefore, a necessary condition for equilibrium is

$$v_i = \arg max_z \{[v_i - b(z)]F(z)^{N-1}\} \qquad (2)$$

for all v_i, . The first order condition for this problem[3] defines a differential equation in v_i:

$$b'(v_i) = \frac{[v_i - b(v_i)](N-1)f(v_i)}{F(v_i)} \tag{3}$$

whose solution[4] is

$$b(v_i) = \int_0^{v_i} \frac{x(N-1)f(x)F(x)^{N-2}dx}{F(v_i)^{N-1}} \tag{4}$$

Notice that this is the expected value of the first order statistic[5] of N-1 draws of the random variable with c.d.f. F conditional on all of them being below v_i.

Obtaining the equilibrium $\beta(v_i)$ for an all-pay auction is a very similar exercise. Indeed, we need only substituting

$$v_i = \arg max_z \quad \{v_i \; F(z)^{N-1} - \beta(z)\} \tag{5}$$

for (2) above to obtain as a solution to the first order conditions to this problem

$$\beta(v_i) = \int_0^{v_i} x(N-1)f(x)F(x)^{N-2}dx. \tag{6}$$

With respect to the oral auctions, we don't need much to obtain equilibrium strategies. Indeed, the Dutch auction is strategically equivalent to the first price auction: one can think of a strategy there as a stopping price, the price at which the buyer decides to jump in and accept the price. The pay-off is exactly given by (1) above for an stopping price $b(v_i)$. On the other hand, the English auction has a dominant strategy in this setting: precisely, accepting any price the seller names as far as it is below the buyer's own valuation. Then, the buyer with highest valuation wins and pays an amount equal to the second highest: the same result as in the second price auction.

Now that we have a prediction of how buyers will behave in the different auctions, we can analyze the expected outcomes of these auctions. That is, we can answer questions

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[3] We are assuming differentiability of $b(v_i)$, but note that we were already assuming monotonicity and therefore, since $b(v_i)$ is obviouly bounded, differentiability almost everywhere.

[4] We are using the initial condition that the bid of a buyer with valuation 0 is 0. Indeed, it makes no sense for her to bid more, and bidding less would not make part of an equilibrium: note that in any monotone equilibrium a buyer with valuation 0 never wins the auction and therefore makes zero rents. Now, if $b(0) < 0$ then the buyer could benefit from increasing its bid to, say $b(0)/2$, now obtaining the object with some probability at a price lower than her valutuation. That is, obtaining some positive rents.

[5] The n^{th} order statistic is simply the n^{th} highest realization.

such as what auction gives higher expected revenues to the seller, or higher rents to the buyers, and which is more efficient. For instance, the answer to this last question is that all of them are equally efficient, in fact, fully efficient. Indeed, the fact that the bidding functions in all of them are strictly monotone implies that in all of them the buyer with highest valuation obtains the good. But this is the only condition for efficiency: all payments are transfers from one agent to another, and therefore the total surplus generated by the trade is simply the difference between the winning buyer's valuation and the seller's one.

Turning to the seller's revenue, in a second price auction the winner pays the second highest bid, and then the seller's revenue is simply the second highest valuation, that is, the second order statistic of N draws from the random variable with c.d.f. F. In a first price auction, the revenue is the bid of the highest valuation bidder. Remember that this bid is the expected value of the highest of N-1 draws below her valuation. That is, in expected terms, we are talking about the expected value of the second order statistic of N draws again! Indeed, the expected value of the bid tended by the buyer with highest valuation is

$$\begin{aligned}&\int_0^1 b(v)Nf(v)F(v)^{N-1}dv = \int_0^1\int_0^v x(N-1)f(x)F(x)^{N-2}dx\, Nf(v)d(v) = \\ &\int_0^1\int_x^1 Nf(v)dv\, x(N-1)f(x)F(x)^{N-2}dx = \int_0^1 xN(N-1)f(x)[1-F(x)]F(x)^{N-2}dx,\end{aligned} \tag{7}$$

which is nothing[6] but the expected value of the second order statistic of N realizations of F.

Similarly, we can check that the all-pay auction gives the same revenues to the seller, in expected terms. We can use an alternative approach. Indeed, by definition, the seller's revenues are simply the difference between the surplus generated by the trade and the buyers' rents. Now, integrating by parts in (6) and substituting the new expression for β(v) in (5), one obtains that the rents a buyer with valuation v expects in an all-pay auction are

$$\int_0^v F(x)^{N-1}dv, \tag{8}$$

which is exactly what one obtains integrating by parts in (4) and substituting in (1). Thus, each buyer expects the same rents for each possible valuation in both first price and all-pay

[6] We have used a little trick quite common in auction theory, to obtain the second equality above. This is simply to change the order of integration. Indeed, the right hand side of the first equality is simply the integral of a function in v and x in the triangle obtained letting v range from 0 to 1 and then x range from 0 to v. This is the same triangle that one obtains letting x range from 0 to 1 and then v range from x to 1.

auctions. Therefore, both the buyer's rents and the seller's revenues are the same under both trading mechanisms even conditional on the valuation of the winning bid. And these identities can be extended as easily to the second price auction. The three of them are, in this sense, equivalent[7].

This may seem a surprising result. In the face of it, we should wonder what is there that explains the coincidence. Also, one can ambition a little more and try to find better ways (from the seller's point of view) of selling the good. Perhaps, why not, the best way (Myerson (1981), Riley and Samuelson (1981)). This last question seems out of reach, given that the set of possible trading mechanisms (think of any exotic set of rules that the seller can use to carry out the transaction) is infinite and, worse, has no structure at all. But there is a way to tackle this question: invert the problem. Instead of considering one by one the possible mechanisms, then analyze the behavior, and finally obtain the results, one can look at the set of conceivable results, first. Then one can ask which of these results can be obtained. There is hope in this, since the set of conceivable results does have some structure. Indeed, any result of a trading mechanism is a rule that assigns the object to some buyer and assigns payments to each, all as a function of their valuations. Call these rules allocation rules. Now, one can hope that the set of all the feasible allocation rules, among the conceivable, still preserves some simple structure[8].

Thus, an allocation rule is a set of functions $\{\rho_i(v),\chi_i(v)\}_{i=1,2,\dots,N}$, where the first is the payment i will make to the seller if v is the vector of valuations and the second is the probability that i obtains the good in this case. Of course, for any v, $\Sigma_i\, \chi_i(v) \leq 1$, where the possibility of inequality comes from the fact that the seller can leave the object unsold. For most of this section, we will only need a summary of these functions. So, let $p_i(v_i) = E_{v_{-i}}[\rho_i(v)]$ and $x_i(v_i) = E_{v_{-i}}[\chi_i(v)]$, that is, respectively, how much buyer i expects to pay and the probability that she obtains the object given what she knows when entering the trade: her valuation.

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[7] The realized price, however, is different. Given the buyer's valuation, how much she pays is deterministic in the all-pay auction and random in the other two, a randomness that, once one conditions on winning the auction, disappears in the first price auction and persists in the second price auction.

[8] For the reader familiar to mechanism design, we will of course be using the logic behind the revelation principle (Myerson (1979) is the classic reference).

The goal for the seller is to maximize the expected revenue, that is,

$$\sum_i E_{v_i}\left[p_i(v_i)\right]. \tag{9}$$

Now, there are two constraints (at least) that an allocation rule has to satisfy in order to be feasible (implementable). First, no buyer should be expected to enter into any trading mechanism if she expects to loose by doing it, and therefore (assuming the opportunity cost for the buyer is zero), we won't have an implementable allocation rule unless $p_i(v_i) \geq 0$ vor all v_i. This is known as the individual rationality constraint. Second, in any trading mechanism there is no way to impose the choice of actions to a player. In particular, a buyer with certain valuation v_i can always choose to act as the allocation rule expects a buyer with some other valuation, say z, to act. Therefore, one can expect to implement an allocation rule (through a mechanism) only if one is certain that each buyer has the incentive to behave as predicted for her valuation. That is, if for all v_i

$$v_i = \arg max_z \{x_i(z)v_i - p_i(z)\}, \tag{10}$$

whose first order condition is the differential equation

$$x_i'(v_i)v_i = p_i'(v_i). \tag{11}$$

Integrating in both sides, we obtain the solution

$$p_i(v_i) = \int_0^{v_i} z\, x_i'(z)dz + p_i(0). \tag{12}$$

Equation (12) already implies the equivalence of all five types of auctions analyzed above. It also constitutes the essence of what is known as "revenue equivalence theorem"(Myerson [1981]). This theorem says that any two auctions that assign the object in the same way (that is with common functions $x_i()$), should have the same payment functions $p_i()$ too, provided these last functions coincide at least at one point (at $p_i(0)$, for instance). In all auctions analyzed above, the object was assigned to the buyer with highest valuation because all had symmetric, monotone equilibria. Also, a buyer with valuation 0 could make sure to pay 0 too. Thus, expected payments for any buyer with any given valuation were the same in all five auctions, and the seller expected the same revenues.

It is useful to spend a little more time analyzing this result. Integrating by parts in (12), we get

$$x_i(v_i)v_i - p_i(v_i) = [x_i(0)0 - p_i(0)] + \int_0^{v_i} x_i(z)dz. \quad (13)$$

The left hand side in (13) represents the rents for buyer i with valuation v_i. These (informational) rents are monotone in the valuation in a way which is standard in the literature on adverse selection: A buyer with valuation $v_i + \Delta$ can always pretend to be of valuation v_i. This would mean the same payment $p(v_i)$ but a value for the buyer equal to $(v_i + \Delta)\, x_i(v_i)$, instead of $v_i\, x_i(v_i)$. Therefore, the rents for the buyer with valuation $(v_i + \Delta)$ should be at least $\Delta\, x_i(v_i)$ higher than those for a buyer with valuation v_i. But a buyer with valuation v_i can also imitate a buyer with valuation $v_i + \Delta$ and therefore, by a similar argument, the difference in rents cannot exceed $\Delta\, x_i(v_i)$, for infinitesimal[9] Δ. That is what equation (13) states.

As we will see in later sections, this direct relationship of rents and allocation of the object is a quite powerful tool when comparing different auctions or searching for optimal ones. For the case we are analyzing, it allows us to find what will be the best trading mechanism. The crucial part is that we can basically disregard the functions $p_i(v_i)$ to concentrate in the way the mechanism allocates the object to potential buyers: given a vector of functions[10] $x_i(v_i)$ (and a set of initial conditions $p_i(0)$), there are payment functions $p_i(v_i)$ (virtually unique) which are compatible with that vector. And this is all we need now. Indeed, substituting (13) in (9), the seller's problem can be rewritten as[11]

[9] We need Δ to be infinitesimal so as to be able to disregard the differences in probability of obtaining the object for the two types considered.

[10] This should be read as a necessary condition for the trading mechanism. If we add that $x_i(v_i)$ are monotone then we have necessary and sufficient conditions for implementation, in fact.

[11] The first equality is obtained by using again the little trick commented in footnote 4: changing the order of variables in integrating over a triangle.

$$
\begin{aligned}
&Max\sum_i\left\{\int_0^1\left(x_i(v_i)v_i-\int_0^{v_i}x_i(z)dz\right)f(v_i)dv_i+\left[p_i(0)-x_i(0)0\right]\right\}=\\
&Max\sum_i\left\{\int_0^1\left(x_i(v_i)v_if(v_i)-\int_{v_i}^1 x_i(v_i)\,f(z)\,dz\right)dv_i+\left[p_i(0)-x_i(0)0\right]\right\}=\\
&Max\sum_i\left\{\int_0^1\left[v_i-\frac{1-F(v_i)}{f(v_i)}\right]x_i(v_i)f(v_i)dv_i+\left[p_i(0)-x_i(0)0\right]\right\}=\\
&Max\left\{\int_v\sum_i\left[v_i-\frac{1-F(v_i)}{f(v_i)}\right]\chi_i(\upsilon)\Pi_{i=1}^N f(v_i)dv+\left[p_i(0)-x_i(0)0\right]\right\}
\end{aligned}
\qquad (14)
$$

Now, notice that an auction that solves this problem necessarily leaves zero rents to a buyer with valuation 0: in fact, since we have seen that rents need be monotone, this is the only point at which the participation (or individual rationality) constraint need be considered. Finally, under the assumption that the "virtual valuation", defined as

$$J(v_i)=v_i-\frac{1-F(v_i)}{f(v_i)} \qquad (15)$$

is monotone in v_i, which is known as the "regularity assumption", the solution to (14) is quite straightforward: for each value of the valuation vector $\mathbf{v}$, the summation inside the integral is maximized making $\chi_i(\mathbf{v}) = 1$ for buyer i with highest value of J(), which in the regular case means assigning the object to the buyer with highest valuation v_i, provided it is such that $J(v_i) \geq 0$. That is, whenever the highest valuation is above the value v* defined by

$$J(v^*) = 0, \qquad (16)$$

and leaving the object unsold otherwise. An English or Dutch auction, a first or second price auction, each of them with reserve price (minimum acceptable bid) of v* is then an optimal auction in the regular case[12]. Notice that the reserve price that defines the optimal auction does not depend on N. It coincides with the best " take it or leave" offer for a seller that faces a unique potential buyer.

This last result is also familiar to the literature on incentives. From our discussion on informational rents above, it should be clear that a way to reduce these rents is to manipulate the allocation (probabilities) of the object. This has a drawback, since that also

[12] In the non-regular case, assigning the good to the buyer with highest value of the function J() is different from assigning it to the buyer with highest value, and then that goes against monotonicity of the functions x(). The optimal auction is then a more complex one.

reduces the surplus from the trade. The most successful way to manipulate the allocation is to reduce $x_i(v_i)$ for low values of v_i. Indeed, this has the largest impact on revenues (reducing $x_i(v_i)$ affects the rents of all buyers with valuations above v_i) and the lowest on surplus (the gain from trade for low values of v_i is low too). The value v^* can be defined as the one at which these two effects exactly balance.

3.- Extensions

In this section, we study how the conclusions we have reached above change when we modify some of the assumptions of our basic model. In particular, we study the effects of endogenous entry, risk aversion, asymmetric buyers, and correlated values.

3.1: Entry

Up to now, we have assumed that the number of buyers was exogenously determined. In many cases, however, how many buyers a seller can attract depends on the trading mechanism she announces. Indeed, in case there is any cost of participating in the auction, a potential buyer has to weigh those costs against the rents she can expect if taking part in the auction. As we have seen in the previous section, these rents, and therefore the entry decision, will depend on the auction rules.

But, what are those entry costs? Taking part in an auction may imply costs related to actually "playing the game" (transportation, bid preparation,...), or costs related to even learning about the object (inspection, ...). That is, one could distinguish between costs the buyer has to incur before even knowing her valuation and costs incurred after learning the valuation but before submitting the bid. As we will see, some of the conclusions in the previous section will change when we consider any of these types of entry costs.

Thus, assume that, before learning her valuation, a bidder has to decide whether to incur a cost c that buys her this information and any other resources needed to participate in the auction. We ask the question of what would then be the optimal reserve price r in a first price auction[13]. One thing to notice is that, after all the buyers have taken their decisions and the bidding process starts (and assuming for convenience that everybody observes how many buyers have incurred the cost c and then are able to bid), the analysis in the previous

[13] Everything would be the same assuming English, Dutch, or second price auctions.

section still applies: In particular, (3) still defines the equilibrium, symmetric, monotone bidding function (except that this time the boundary condition is b(r) = r). Then, (13) defines the gross (that is, before deducting the entry cost c) buyers' rents, with

$$x_i(v_i) = \begin{cases} 0 & \text{if } v_i < r \\ F(v_i)^{n-1} & \text{if } v_i \geq r \end{cases} \tag{16}$$

where n is the number of buyers who have entered (paid the cost c). And therefore, the gross rents a buyer expects from entering before knowing her valuation but knowing that some other n-1 buyers will enter too, are

$$E\Pi(r;n) = \int_r^1 \int_r^v F(z)^{n-1} dz\, f(v) dv = \int_r^1 \int_z^1 F(z)^{n-1} f(v) dv dz = \int_r^1 [1 - F(z)] F(z)^{n-1} dz. \tag{17}$$

Notice that these expected rents are decreasing in n. In particular, for N large enough, if every potential buyer enters the auction (n=N), the gross expected rents will be lower than the entry cost c. In this case, in equilibrium not all buyers will enter. Thus, any pure strategy equilibrium (in entry decisions) will have to be asymmetric in that some n buyers will enter and some N-n won't, even though ex-ante all of them are symmetric. And any symmetric equilibrium will have to be in mixed strategies in that each buyer will enter with some probability q<1. Both types of equilibria have been studied in the literature[14]. Consider the first. Disregarding the integer problem[15], equilibrium entry n will be defined by $E\Pi(r;n) = c$. That is, entry will occur until the point net rents are driven down to zero. But then the seller's revenues are simply the expected surplus from trade minus entry costs, nc.

This immediately shows that the optimal reserve price is zero. Indeed, take any reserve price r and the induced entry level n. Now, consider as an alternative auctioning the object without reserve price but charging an entry fee (a price a buyer has to pay the seller before submitting the bid) $e = E\Pi(0;n) - c$. It is immediate to check that n is still the equilibrium entry level. Then the revenue for the seller is higher since the surplus is higher (the object is always sold) and the entry costs are the same as before.

Moreover, this also shows that the optimal entry level coincides with the efficient

[14] See McAfee and McMillan (1987) for the first type and Samuelson (1985) for the second.

[15] In general, this integer problem will matter, but only to the point that some marginal entry fees will be needed in order to extract marginal rents: with zero reserve price and given that entry must be in integer numbers, buyers will expect some positive rents that can be extracted by an entry fee equal to those rents.

one. Indeed, since entry drives buyers' rents to zero, the seller's revenues coincide with the net surplus: surplus from trade minus entry costs. What is this efficient entry level? Entry of a new buyer contributes to the net surplus in that it represent an additional opportunity for high valuations, but it also detracts from this surplus in that it represents an additional entry cost of c. Thus, entry should continue until the two effects cancel, and not after that. So, again ignoring integer problems, n* is such that the last entry increases surplus from trade by an amount equal to c. That is, n* is such that the difference in surplus when entry is n* and when entry is n*-1 equals c. That is,

$$c = \int_0^1 n^* x f(x) F(x)^{n^*-1} dx - \int_0^1 (n^*-1) x f(x) F(x)^{n^*-2} dx = \int_0^1 [1-F(x)] F(x)^{n^*-1} dx, \quad (18)$$

where the last equality is obtained integrating by parts. That is, according to (17), n* is the same that solves $E\Pi(0;n^*) = c$, i.e., the level of entry that is obtained with a first price auction with no entry fee and no reserve price!

The result is in sharp contrast with the one obtained in Section 2. Indeed, here rent extraction is not an issue, since endogenous entry pushes these rents to zero. The interesting point is that buyers' incentives to enter are exactly the correct, from the social as well as from the seller's point of view. Put it another way, when a seller considers whether to have higher competition that dissipates buyers' rents or to award to some buyers some "monopoly rents" that the seller then extracts, say, using entry fees, the decision should always favor the first alternative.[16] [17]

If we consider the symmetric, mixed strategy equilibrium instead, the conclusions are the same: reserve prices are a dominated tool, efficient and optimal entry coincide, and this entry is the one obtained with no entry fee, that is, using a standard first price auction.

Things are slightly different if buyers know their valuation before they have to decide whether to enter or not. In this case, a buyer can condition her entry decision to her valuation. Then, rents cannot be completely extracted since buyers are better informed than

[16] Bulow and Klemperer (1996), in a very related result, show that having one more buyer in one auction is better that not allowing this extra buyer to take part in the auction but making hima take it or leave it offer after the auction as an alternative to selling to the winner in the auction at the hammered down price.

[17] This conclussion hinges on the private values assumption. Indeed, if we assumed common values, as later in this same section, the efficient and optimal policies could be to curtail entry. Indeed, in this case the surplus from trade does not depend on N but the entry costs still do (see French and McCormick [1981]).

the seller <u>before</u> joining the game. Yet some sort of zero rent condition will determine entry here too. Indeed, one can expect buyers to enter if their valuations are high enough (expecting high enough rents upon entry) to recover the entry cost, and therefore one can expect that equilibrium entry will be characterized by some cut-off level v*, the valuation of a buyer who expects to exactly break even, that is, expects gross rents exactly equal to c. Then buyers will enter if and only if their valuations are above v*. If bidding is monotone after entry, a buyer with valuation v* won't bid in equilibrium more that the reserve price r, since she wins the auction only in case nobody else enters. Then

$$(v^*-r)F(v^*)^{N-1}=c \tag{19}$$

defines this cut-off point v* and entry (now understood as minimum valuation at which a buyer decides to participate) as a function of r. For buyers with valuation above v*, (13) still defines the incentive constraint and the (net) rents are given by (17) with only substituting v* for r and N for n.

Notice that v* is monotone in r. Thus, choosing optimal r is equivalent to choosing the entry level v*. Then the optimal auction will maximize the difference between net surplus and rents, that is

$$\int_{v^*}^{1} Nzf(z)F(z)^{N-1}dz - N\int_{v^*}^{1}[1-F(z)]F(z)^{N-1}dz - NF(v^*)^{N-1}c\,, \tag{20}$$

where the first term represents gross surplus from trade, the second is the buyers' rents and the third the entry costs. The first order conditions for maximization of (20) with respect to v* has the following first order condition:

$$N[v^*f(v^*)F(v^*)^{N-1}-[1-F(v^*)]F(v^*)^{N-1}-cf(v^*)]=0\,, \tag{21}$$

which simplifies to:

$$v^*-c=\frac{1-F(v^*)}{f(v^*)}. \tag{22}$$

Notice that if c = 0 we are back to (16). As c increases so does v*. However, from (19) and for given v*, r is decreasing in c. Then, the effect of entry costs on optimal r is ambiguous in general. However, analyzing (21) we can guarantee that $v^*\to 1$ as $N\to\infty$ (with very high competition, only the very high valuation bidders have any chance to win the object and then have some positive rent, even without any reserve price), but $F(v^*)^{N-1}\to c$, which by

(19) implies that r $\rightarrow$ 0! That is, for N large enough we conclude again that setting a positive reserve price is not in the seller's interest.

Also, as in the former case, one can easily check that efficient entry occurs when r is set to zero. That is, once again, buyers have the right incentives to enter.

There is one more difference that the endogenous entry introduces in auction design. Indeed, in this section we have been talking about optimal first price auctions. This was for good reasons. Indeed, the auctions we have analyzed are not optimal auctions in general. The reason is that the indirect approach taken in the previous section does not guarantee now that we are considering all possible auctions. It does not consider, for instance, auctions in which whether some buyer buys information or not depends on the information obtained by another buyer who did buy hers[18]. How could the seller arrange things so that this type of contingent entry takes place? The answer takes us back to the use of reserve prices! Indeed, we have seen that a reserve price is not a very adequate instrument (and is never better than entry fees) for rent extraction when entry is endogenous. However, precisely in this case, reserve prices are a very convenient way to manage entry. Thus, in a recent paper (Burguet and Sákovics (1996)) we have shown that the seller can improve upon the static, optimal first price auction by announcing a (first price) auction with a reserve price that, in case no buyer submits bids, would be followed by an auction with zero reserve price. In such a two stage auction, fewer buyers will enter in the first stage. However, if their valuation is low these buyers won't bid in that stage. Then buyers that did not find it profitable to buy information in the first stage now can infer that the buyers that did enter have low valuation, which makes entry more attractive in the second stage. Ex post, what the seller has implemented is restricted entry (and entry costs) if some buyers have high valuation and higher entry in case they don't. The reserve price does not necessarily mean an inefficiency (the object is always sold), but an instrument to balance entry costs with the search for higher valuations.

3.2 Asymmetry

We can now return to the basic model and drop this time the symmetry

[18] For the reader familiar with implementation theory, the revelation principle does not apply here.

assumption. That is, we can assume that buyers are asymmetric even from an ex-ante view point. That simply means that each v_i is the realization of a random variable with distinct distribution F_i and density f_i. This change in our assumptions has little conceptual implications. In particular, we could take the indirect approach used in the previous section with no change: in fact we didn't really used the symmetry assumption from equation (9) to equation (14). The heart of that indirect approach, the incentive compatibility constraint on trading mechanisms, according to which rents were virtually determined by the fact that a buyer can always pretend to have some other valuation, considered each buyer in isolation. Then, the revenue equivalence theorem still holds in the sense that any two mechanisms that have common functions $x_i()$ (and give same rents to buyers with zero valuation) are equivalent. Also, the solution to the maximization problem (14) will still have the buyer with highest virtual valuation winning the auction, provided this highest virtual valuation is above zero. However, in this case the virtual valuation functions are different for different buyers. That is, $J_i(v_i)$ does depend on i. Then, highest virtual valuation does not mean highest valuation even in the regular case. Also, our standard auctions (even complemented with a reserve price) won't be optimal auctions.

Consider a very simple example with N = 2, were both buyers have valuations distributed uniformly, but v_1 is uniform in [0,1] whereas v_2 is uniform in [0,2]. In this case $J_1(v_1) = 2 v_1 - 1$ and $J_2(v_2) = 2 v_2 - 2$. Then the optimal auction is to assign the object to buyer 1 if $v_1 > max\{ v_2 - 1/2, 1/2\}$, to buyer 2 if $v_2 > max\{v_1 + 1/2, 1\}$, and not selling the object otherwise. How could this be accomplished? By means of a second price auction with reserve price 1 and bonus to buyer 1 of 1/2. That is, if bidding above buyer 2, buyer 1 has to pay buyer 2's bid minus the bonus of 1/2. It is simple to check that bidding the true valuation is still dominant for buyer 2 and bidding the true valuation plus ½ is dominant for buyer 1, which leads to the optimal auction assignment.

This example illustrates the two main points to be learnt about auctions for asymmetric buyers: in general it is in the seller's interest to discriminate in favor of (ex-ante) weaker buyers. This increases the otherwise soft competition that stronger buyers face, which helps reduce their informational rents (see McAfee and McMillan [1989] and Branco [1994], the last for asymmetries coming from the seller's goals). The second point is a negative one: second price auctions, with or without bonuses, are not in general optimal. If the distribution was not uniform, there is no bonus that makes a second price auction replicate the optimal auction. That is unfortunate, since one would like to have optimal auctions that are robust in their design to non observable parameters of the model. In the previous section we had that lucky result, since standard auctions with reserve prices were optimal for a very large class of distribution functions, any number of buyers, etc. That is, rules were invariant to these changes in parameters, whereas we don't have that invariance now. On the other hand, this makes the optimal auction approach less attractive, and one may prefer to know what is an optimal auction among a set of simple, invariant ones. For instance, which one, first or second price auctions (respectively, Dutch or English auctions) gives higher revenues to the seller. Unfortunately, even this simple question does not have a definite answer. As Maskin and Riley (1985) have shown, the answer depends again on the particular distributions F_i.[19]

3.3: Risk aversion

We now drop the risk neutrality assumption and assume instead that buyer i has utility $u_i(v_i\text{-}p)$ if she obtains the object and pays p, and utility $u_i(\text{-}p)$ is she does not get the object and pays p, where we assume $u_i()$ to be concave[20], and the rest of assumptions are as in the Section 2. Then buyers won't be indifferent between two auctions that allow them the same expected rents but differ in the uncertainties they represent. There are two types of uncertainties a buyer can face when participating in an auction. First, her utility, may be different when winning the object and when loosing it. Second, even conditional on winning the object (or loosing it), her payments, and then again her utility, may still be subject to some randomness (as in the second price auction).

Under risk aversion, buyers prefer auctions that imply lower risks, in the sense that the uncertainty about their utility is lower. Put in other words, buyers are less willing to pay for participating in a high uncertainty lottery for the object. This suggests that it is in a seller's interest to offer auctions that insure buyers as much as possible. However, when

[19] If the question was efficiency, the aswer would be clear: a second price auction does assign the object efficiently, whereas first price auctions don't, in general.

[20] Much more general types of utility functions could be considered. See Maskin and Riley (1984).

buyers are risk averse, risk may be an interesting screening device.

Indeed, consider again any trading mechanism described as in Section 2, by way of the functions $x_i()$ and $p_i()$. One could consider these functions as a menu of (linear) contracts offered to the buyer: by choosing to report z as her valuation, the buyer chooses the contract $\{x_i(z), p_i(z)\}$. Different types of buyers obtain different expected values from different contracts: $x_i(z)\ v_i - p_i(z)$. That is the way the seller screened the buyer type in Section 2. This need to screen buyers was the source of informational rents, and was represented by the incentive compatibility constraint. When buyers are risk averse, however, expected rents are not the only concern for the buyers. Then, the seller may increase $p_i()$ for high v_i and still make sure that a buyer with that valuation prefers not to report a lower valuation z by introducing more risk in the contract designed for z (that is, more variability in $p_i(z)$ or higher difference in the utility obtained when winning and when loosing if the true valuation is v_i).

Maskin and Riley (1984) show that one of the types of uncertainty we mentioned above, the one about payments for given allocation of the good, never pays. However, they also show that the other type of uncertainty is indeed a useful tool for screening that helps reducing informational rents. That is, an optimal auction would not completely insure buyers. In particular, buyers' payment conditional on winning and payment conditional on loosing should be deterministic, but buyers should prefer winning the object than loosing (except for the highest possible valuation). The difference in utility (risk) should be decreasing in the valuation. This last result, in fact, implies that high valuation buyers should be paid (subsidized) in case they don't win the auction. [21]

To illustrate these points, let us compare first price and second price auctions in our case[22]. First, note that in a second price auction $B(v_i) = v_i$ is still dominant strategy under risk aversion. Indeed, whatever risk associated, a buyer prefers to win if the price is below her valuation and vice versa. Then, the expected payment by a winning buyer with valuation v_i is still given by the right hand side of (4) above. The derivative of this expected payment with respect to v_i is equal to

[21] Using subsidies as a screening device may also be interesting when faced with dynamic, endogenous entry (see Burguet [1996]).
[22] We follow here Maskin and Riley (1984).

$$\frac{dp_i(v_i|winning)}{v_i} = \frac{(N-1)f(v_i)}{F(v_i)}[v_i - p_i(v_i|winning)]. \tag{23}$$

On the other hand, a symmetric, monotone equilibrium bidding function for a first price auction should satisfy

$$v_i = \arg max_z \quad F(z)^{N-1} u_i(v_i - b(z)), \tag{24}$$

where we are assuming (for economy in notation) that u(0)=0. Then, the first order condition for this problem implies that

$$b'(v_i) = \frac{(N-1)f(v_i)}{F(v_i)} \frac{u(v_i - b(v_i))}{u'(v_i - b(v_i))} \tag{25}$$

Now, notice that in a first price auction $b(v_i) = p_i(v_i|\text{ winning})$, and notice that (if a symmetric, monotone equilibrium exists, which in our case is guaranteed; see Maskin and Riley (1984)), the same bidder wins. Then, if we can show that $p_i(v_i|\text{ winning})$ is always higher in a first price auction than in a second price auction, we would conclude that the seller expects higher revenues (for any realization of the valuations vector v) in the first[23]. This illustrates the point that introducing uncertainty beyond the point of different utility when winning and when loosing is counterproductive.

Now, to show that $p_i(v_i|\text{ winning})$ is always higher in a first price auction, we use a little lemma (see Milgrom and Weber (1982), lemma 2, of which the present one is a slightly reformulated version):

Lemma: let g and h be two differentiable functions defined in [0,1] such that (i) g(0)=h(0) and (ii) g(x) = h(x) implies that g'(x) > h'(x). Then g(x) > h(x) for all x in (0,1].

Applying this to our case, one can notice that both first price and second price auctions imply zero expected payments for a winning bidder with valuation 0. Second, assume that, for some value v_i, $p_i(v_i|\text{ winning})$ coincides in both first price and second price auctions. That means that even in the second price auction $p_i(v_i|\text{ winning}) = b(v_i)$, and then at that point

[23] Actually, this is also enough to conclude that the seller prefers the first price auction even if she herself is risk averse. Indeed, the first price auction is the same lottery with respect to who wins the auction, but the payments in the first price auction given who wins are not only higher in expected terms but also riskless. For a study of seller's preferences over different designs when the seller is reisk averse, see Waehrer, Harstad, and Rothkopf (1998).

$$\frac{dp_i(v_i|winning)}{v_i} = \frac{(N-1)f(v_i)}{F(v_i)}[v_i - b(v_i)] \quad (26)$$

for the second price auction. Then, we only need to show that at such a point

$$v_i - b(v_i) < \frac{u_i(v_i - b(v_i))}{u'_i(v_i - b(v_i))}, \quad (27)$$

which follows from the facts that (i) considered as a function of b, the derivative of the left hand side is -1 whereas the derivative of the right hand side is $-1 + \frac{u_i''u_i}{(u_i')^2} < -1$, and (ii), at b = v_i the right hand side is equal to the left hand side (and equal to zero), but at any point v_i, $b(v_i) < v_i$.

As we mentioned before, this could be (wrongly) extended to conclude that the lowest the risk for the buyers the higher the revenues for the seller. To show this wrong, consider an auction that would perfectly insure buyers. Such an auction would have fixed payments for the case the bidder won the auction, call it $d(v_i)$, and fixed payments for the case the bidder lost the auction, call it $a(v_i)$, which would satisfy $v_i - b(v_i) = -a(v_i)$, for all v_i. Then, incentive compatibility would require that

$$v_i = \arg max_z \;\; x_i(z)u_i(v_i - d(z)) + (1 - x_i(z))u_i(-a(z)) \quad (28)$$

whose first order conditions imply $(a' - 1 =)\ d' = 1 - x_i'$. That is, in such an auction the revenue does not depend on the utility function u. In particular, the revenue would be the same if buyers were risk neutral. But under risk neutrality, a second price auction with a reserve price maximizes the (expected) revenues for the seller. Since the revenues in a second price auction is also independent on the buyers' attitude towards risk, we conclude that a perfect insurance auction does not represent more revenues to the seller than a second price auction (with reserve price), which in turn produces less revenues than a first price auction (with reserve prices, extending in a straightforward way our previous comparison).

3.4: Correlated information; non private values

Perhaps the most important assumption we made in Section 2, however, was that information was uncorrelated and that valuations were private. (In fact, in that section valuations and information were the same thing.) If information is not independent, a seller can use the (more informed) inferences a buyer can make about other buyers valuations to both extract this information and even screen that buyer's own information (proposing bets on other buyers' information; see Myerson [1981]). In some circumstances, the seller can extract all the surplus in this way (see Crémer and McLean [1988]). Another implication of this is that the revenue equivalence results do not hold. And for reasons very similar to the ones discussed when we dealt with risk aversion: now the seller has an additional screening device, and two auctions that use the probability instrument in the same way may still differ in their use of the inference instrument. We will see this later in this section.

With respect to private values, the assumption is usually inappropriate in cases such as the auctioning of mineral rights, goods with resale market, like securities or real state, etc. In fact, in those cases, assuming that the valuation of the good is the same for all buyers is probably a better assumption. But if the value of the good is the same for every potential buyer, what is there that makes the problem interesting? The answer has to do with residual uncertainty about this common valuation at the bidding time. Indeed, when firms bid for mineral rights on a tract of land there is still a lot of uncertainty about the amount of mineral that can be extracted, or the future prices for that mineral, or the costs involved. Moreover, different bidders may have different estimates of this common value. That is, they may have different information about this common value. Also, they should take into account how the auction processes this information and, therefore, what inferences can be made about others' information.

So, let us now assume that each buyer i receives some information about the common value v of some object. This information is formalized by a signal s_i which is correlated with v. The vector $(v, s_1, s_2, \ldots, s_N)$ is distributed according to some probability function, which is common knowledge. We assume that higher signals mean better news with respect to v and with respect to all other signals[24]. An example would have $v = s_1 + s_2 + \ldots + s_N$, with all s_i being independent draws of some random variable. This would be a case of common values with independent information.

[24] More formally, we will always assume that all random variables are *affiliated*, a generalization of positive correlation which roughly extends this to any subsets in the domains of the variables (see Milgrom and Weber (1982)).

One of the most popular phenomena in common values auctions is the so-called *winner's curse*. Consider the simple case we have just mentioned in which the value is the sum of all signals, and assume these signals are uniformly distributed in [0,1], and N = 5. One could still hope that what we learnt about second price auctions in Section 2 applies here. That is, an equilibrium bidding strategy would be to bid $E[v| s_i]$, the expected value of v conditional on the information the buyer has. In our example, that means $B(s_i) = 2 + s_i$. But if everybody uses that strategy, then buyer i will win the auction only when her signal is highest. That is, when all other signals are below s_i, in which case the expected value of the object $E[v| s_i; \forall j, s_j \leq s_i] = 3\ s_i$. The price expected by buyer i when she wins is, on the other hand, $2 + (4/5)\ s_i$. Thus, if $s_i = \frac{1}{2}$, buyer i expects to pay 2.4 when she wins for an object that, precisely conditional on her winning, has an expected value of only 1.5! Thus, bidding according to the proposed strategy does not seem dominant, but rather disastrous.

The reason is that winning conveys itself information about others' signals. When a buyer decides her bid she has to take into account that her bid is relevant only for the case she wins. If the equilibrium bidding strategy is increasing in the signal, then winning means that her information has been the most optimistic. Thus, when bidding buyers should discount in their bids this excessive optimism.

As we mentioned above, two auctions that assign the good in the same fashion don't have to be equivalent from the seller's view point. To illustrate this point, we will compare first price and second price auctions again. So let us return to our second price auction. We have seen that $B(s_i) = E[v| s_i]$ is not a dominant strategy. In fact, it is not hard to see that there could be no dominant strategy equilibria in the common values case: winning always conveys information about others' signals, but what information exactly will depend on the way others use theirs to form their bids. What will then be a Bayesian equilibrium bidding strategy? The answer is

$$B(s_i) = E[v / max_{j \neq i}\ s_j = s_i,\ s_i] \qquad (29)$$

(in our example above, $B(s_i) = 3 \times s_i$). Indeed, in a second price auction, the bid is the highest price a buyer is willing to pay for the object. Now, assume that all bidders but i bid according to (29) and buyer i has a signal s_i. Is buyer i willing to accept a price $p < B(s_i)$? First we have to answer the question of what does it mean that the price (the highest rival bid) is p. Given that all others are using (29), which is increasing in the signal, this means that the highest rival signal, call it s*, is lower than s_i. In particular

$$p = b(s^*) = E[v/max_{j\neq i}\, s_j = s^*, s^*] < E[v/max_{j\neq i}\, s_j = s^*\,,\, s_i] \qquad (30)$$

(the last inequality due to the fact that v is stochastically increasing in s_i). But the right hand side of the inequality is the expected value of v given that i has signal s_i and the highest rival bid is p. That is, buyer i is indeed willing to accept that price p. A similar argument would show that buyer i would not be willing to take any price above $B(s_i)$, which means that this is indeed an equilibrium bidding strategy.

We could now compute the equilibrium bidding strategy in a first price auction (see Milgrom and Weber (1982)), but we won't even need this to conclude that a first price auction means lower revenue to the seller. The way to do it is to come back to our indirect approach of Section 2. Indeed, for any mechanism $\{\rho_i(\mathbf{s}),\chi_i(\mathbf{s})\}_{\,i=1,2,\ldots,N}$, where now the allocation and payments should depend on the vector of signals $\mathbf{s} = (s_1,s_2,\ldots,s_N)$, rather than valuations, define again the functions $p_i(z;\, s_i) = E_{\mathbf{s}_{-i}}\left[\rho_i(\mathbf{s}_{-i}, z)\,\middle|\, s_i\right]$ and $x_i(z;s_i) = E_{\mathbf{s}_{-i}}\left[\chi_i(\mathbf{s}_{-i}, z)\middle| s_i\right]$. Remember that we use this functions to analyze the decision problem of a buyer, where she can decide how to behave (what signal she will pretend to have), but she knows what signal she really has. In the independent information case, this last piece of information was not important when computing the expected payment and probability of winning, since these data depended only on her behavior. Now, however, the probabilities that she assigns to others' signals may depend on her true signal, and therefore expected payments and probability of winning depend on both "reported signal" (z) and true one. Thus, $x_i(z;\, s_i)$ is the probability that i obtains the good if she behaves as having observed signal z but has in fact observed signal s_i, and similarly for $p_i(z;\, s_i)$. We will also need to define what now plays the part of i's valuation. Thus, let

$$R_i(z;\, s_i) = E[v/max_{j\neq i}\, s_j = z,\, s_i]. \qquad (31)$$

Then, for any trading auction i's incentive compatibility constraint can be written as

$$s_i = arg\, max_z\, \{R_i(z;\, s_i)\; x_i(z;\, s_i)\; - p_i(z;\, s_i)\} \qquad (32)$$

with first order condition

$$0 = R_i(s_i;\, s_i)\; x_{i(1)}(s_i;\, s_i) + R_{i(2)}(s_i;\, s_i)\; x_i(s_i;\, s_i) - p_{i(1)}(z;\, s_i) \qquad (33)$$

where the subindex in brackets represents the variable with respect to which we are partially differentiating (e.g., $x_{i(1)}$ represents the partial derivative of x_i with respect to the first argument -z-). Again, the seller is interested in the payment functions evaluated on the diagonal, that is, taking into account that buyers do have incentives to behave according to predictions. So let us concentrate on the study of these functions $p_i(s_i;s_i)$. In particular, we will compare these functions for the first price and second price auctions. We will prove that $p_i(s_i;s_i)$ for the second price auction is everywhere (weakly) above the corresponding function for the first price auction. That is, whatever her valuation, a buyer expects to pay more in the second price auction than in the first price auction. Thus, the expected revenues for the seller are higher in the former than in the latter.

The way to prove this result will be to come back to the lemma in Section 3.3. Indeed, notice that $p_i(0;0) = R_i(0;0)$ in both second price and first price auctions. Then we will prove that whenever $p_i(s_i;s_i)$ coincide for the first price and second price auctions, the slope of this function (along the diagonal) is (weakly) higher for the second price auction than for the first price auction. To this end, first notice that

$$\frac{dp_i(s_i,s_i)}{ds_i} = p_{i(1)}(s_i,s_i) + p_{i(2)}(s_i,s_i). \tag{34}$$

Also, notice that, according to (33), the first term in the right hand side of (34) is common to the first price and second prices auctions. Indeed, the functions $x_i(z;s_i)$ are identical in both auctions, and the function $R_i(z;s_i)$ does not depend on the trading mechanism. Therefore comparing the slopes of the functions $p_i(s_i;s_i)$ is the same as comparing the partial derivative $p_{i(2)}(s_i;s_i)$. Now, in the first price auction

$$p_i(z;s_i) = b(z)\, Pr(z \geq s_j \ \forall j / s_i), \tag{35}$$

where *Pr* stands for probability, whereas in the second price auction

$$p_i(z;s_i) = E[B(max_j\, s_j) / s_i;\ max_j\, s_j \leq z]\, Pr(z \geq s_j \ \forall j / s_i), \tag{36}$$

and then, when $p_i(s_i;s_i)$ coincide for both auctions (which from (35) and (36) implies that $b(z) = E[B(max_j\, s_j) / s_i;\ max_j\, s_j \leq z]$), the difference between $p_{i(2)}(s_i;s_i)$ for the second price auction and the first price auction is simply

$$\Pr(s_i \geq s_j \; \forall j | s_i) \left. \frac{\partial E[B(max_j s_j) | s_i; max_j s_j \leq z]}{\partial s_i} \right|_{z = s_i} \geq 0. \qquad (37)$$

and then we conclude that indeed the function $p_i(s_i;s_i)$ in the second price auction is everywhere above the one for the first price auction. Thus, every buyer expects to pay more in the second price auction, and then the expected revenues for the seller is higher.

Notice that the source of this result is the fact that $p_{i(2)}(s_i;s_i)$ is higher in the second price auction. Also, notice that in obtaining the result we used the fact that the functions $x_i(z;s_i)$ and $R_i(z;s_i)$ where identical in both mechanisms. Now, generalize the definition of function $R_i(z;s_i)$as follows:

$$R_i(z;s_i) = E[v/z \text{ is winning type}, s_i], \qquad (38)$$

Then what we learnt comparing first price and second price auctions can be generalized to one of the key results in auction theory, the "*Linkage Principle*": for any two auctions that assign the good in the same fashion (which implies common $x_i(z;s_i)$ and common inferences $R_i(z;s_i)$) and leave the same rents to a bidder with the lowest possible signal, the revenues for the seller are higher in the auction for which the expected payment of a buyer (for fixed behavior) is more *linked* to her own signal (see Weber (1983)).

It is convenient to underline that the key ingredient for our comparison between auctions (what the linkage principle asked for) was not the fact that the value was common, but the fact that information across agents was correlated. Indeed, even under common values but with independent information (for instance, in our additive example above), first price and second price auctions are equivalent, since then (37) is satisfied with equality. The important feature is that when information is indeed correlated, the fact that buyer i have observed a high signal increases the likelihood of higher signals (and then higher bids) for i's rivals even conditioning on buyer i winning with the same bid!

In fact, everything we are discussing applies to a much more general model, which has the common value and private values models as particular cases. Indeed, one can assume that there is a random vector $(v_1,v_2,\ldots,v_N,s_1,s_2,\ldots,s_N)$ with some probability measure. If this probability measure puts weight only on points where $v_1=v_2=\ldots=v_N$, then we are in the model we have just discussed. If the probability measure puts all its weight on points for which $s_i = v_i$, for all i, then we are in the private values case. If, moreover, the different s_i (and therefore, the different v_i) are independent, we are in the independent private values model. In this more general model, and assuming that all random variables are *affiliated* (see Milgrom and Weber (1982)), the analysis we have just presented extends trivially. In particular, the revenue equivalence result of Section 2 is just a corollary of the linkage principle.

There are many other direct corollaries to the linkage principle. For instance, one can easily conclude from it that the English (oral) auction generates more revenue to the seller than the (sealed bid) second price auction. Indeed, in an oral, ascending auction all buyers adapt their behavior to observed behavior by the rivals. That means that the final price (the "bid" of the last rival to drop from the auction) depends on all signals, not only on this last rival's signal. But that means that the final price is more "linked" to the winner's signal (it is correlated to that signal through the correlation of all signals with the winner's signal) than in a second price auction (where the price is linked to the winner's signal only through its correlation to the second highest bidder's signal). Similar arguments could be invoked to conclude that, if possible, the seller should commit to release any information she might get that is correlated to the buyers' information: that simply increases the correlation among the buyers' signals.

4.- Multiple objects

In this section we extend our analysis to consider the auctioning of multiple units of a homogeneous good. Securities, oil tracts, spectrum rights are all examples of (to a great extent) homogeneous goods that are auctioned to a common set of buyers. Considering multiple units opens a new range of questions (should a seller sell the units simultaneously or in sequence?, what is the optimal size of the lots for sale?) and modeling possibilities (do buyers have a declining inverse demand -willingness to pay- or are synergies among the different units a relevant phenomenon?). All these questions and possibilities were absent or irrelevant when the seller had only one unit to sell.

We start the analysis of multiple unit auctions considering the simplest possible model, in which buyers have positive willingness to pay for only one unit of the good (unit demands). That is, a buyer i is willing to pay up to v_i for a unit of the good, but a second unit has no marginal value to her. This is a very special kind of demand, of course. It may fit situations like the "bidding" for a CEO or some real state auctions. But most important, it will allow us to derive many results as a direct extension of the last two sections.

Indeed, assume all v_i are i.i.d., and assume the seller has K (<N) units of the good. Define the uniform price auction as a generalization of the second price auction, where bidders submit bids, the K highest bidders get one unit of the good and pay the K+1 highest bid. Then it is a simple exercise to check that the strategy $B(v_i) = v_i$, is a dominant strategy in this auction. Also, define the discriminatory auction as a generalization of the first price auction, where the K highest bidders obtain one unit of the good but each pays her bid. Then the same procedure used in Section 2 can be used to conclude that

$$b(v_i) = \frac{\int_0^{v_i} x(N-1)\binom{N-2}{K-1} f(x)F(x)^{N-K-1}[1-F(x)]^{K-1}\,dx}{\int_0^{v_i} (N-1)\binom{N-2}{K-1} f(x)F(x)^{N-K-1}[1-F(x)]^{K-1}\,dx}, \qquad (39)$$

that is, the expected value of the K^{th} order statistic of N-1 realizations of F conditioning on being below v_i, is an equilibrium in this auction. Thus, a buyer with valuation v_i who wins this auction expects to pay the same as this same buyer would pay in the discriminatory auction (the bid, i.e., the valuation, of the K highest among the competitors, provided it is below i's bid, i.e., below v_i). Then these two auctions are equivalent for both the buyers and the seller.

Also, the reason why this revenue equivalence holds is the same we found in Section 2. The incentive compatibility constraint an auction has to satisfy is still given by (10), where now the functions $x_i(v_i)$ are interpreted as the probability of obtaining one good. Also, equation (13) gives the expression for a buyer's informational rents, and implies that two auctions that assign the objects in the same fashion and give the same rents to the marginal buyer are equivalent also from the seller's point of view. Then (14) is the problem that defines the optimal auction and (15) gives the solution: an optimal auction is any one that assigns one object to each of the K highest valuation bidders whose virtual valuation

$J(v_i)$ exceeds zero (Maskin and Riley [1989], Engelbrecht-Wiggans [1988]).[25] For instance, both the uniform price and the discriminatory auctions are optimal when modified by the reserve price v* defined in Section 2.

Consider now auctioning the objects sequentially. That is, the seller auctions one object at a time using at each stage a first price auction or a second price auction. Of course, the bidding behavior described in Section 2 won't be equilibrium in each stage. However, applying the revenue equivalence result above we can conclude that, as far as the K highest valuation buyers end up obtaining one unit of the good (and buyers with zero valuation obtain no rents), the expected revenue for the seller is still the same. Moreover, as far as this is the case, we can compute the expected price in each of the stages without even looking at the equilibrium strategies (see Weber (1983) for the simple form of bidding functions in sequential auctions). The key is to consider the sequence of expected prices from the point of view of the winning bidder in, say, the first auction. Assume that a bidder expects that, conditional on her winning the first auction, the price in the next period is lower that what she would pay in the present period. In this case the bidder would not be behaving optimally this period: she should reduce her bid today and try the cheaper good tomorrow. A similar argument can be made to conclude that the price tomorrow should not be expected to be higher than today, since then bidders today should bid more aggressively.[26] Thus, prices in the sequential auctions should follow a martingale. The ex-ante expected value of this martingale is fixed by the revenue equivalence result we have mentioned above.

Contrary to this theoretical result, prices of sequential auctions of homogeneous goods have been observed to follow too frequently a declining pattern (see, for instance, Ashenfelter (1989), Ashenfelter and Genesove (1992), and Engelbrecht-Wiggans and Kahn (1992)). This is known as the *afternoon effect* or *the declining price anomaly*. What could explain this? Correlation doesn't seem to be a real answer: extending the intuition from the linkage principle discussed in the previous section, future prices should be more linked to a

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[25] The definition of our functions $\chi_i(v)$ is now subject to the constraint that that they take values between 0 and 1 (as before) and, for any v, $\Sigma_i \chi_i(v)$ cannot exceed K (instead of 1).

[26] Here we implicitly use a continuity argument: expected prices tomorrow conditional on prices today are continuous. It is easy to check that this continuity has to be part of the equilibrium features.

buyer's signal or valuation than present ones, since buyers' bids in the future will depend on the inferences made from present bids. Thus, one would rather expect a pattern of increasing prices. Risk aversion, on the other hand, could probably do the job. Indeed, also from the intuition gained in the previous section, a future price is more uncertain than the price today, from today's point of view. For instance, in a first price auction the price is deterministic today conditional on winning, whereas if loosing future prices will depend on today's winning bid, which makes it random. Indeed, McAfee and Vincent (1993) show that any equilibrium in pure strategies for the sequential auction would induce declining prices under risk aversion. Existence of pure strategy equilibria, however, is problematic.

An alternative explanation for declining prices has to do with generalizing the demand functions considered. Black and de Meza (1992) show that downward sloping demands for more than one unit of the good could result in declining prices when the seller offers what is called the "buyer's option": a right for the winner of one round to buy as many units as she wants at the price hammered down in that round. This makes bidders more aggressive in early stages in order to induce higher prices that would dissuade a rival winner from exerting the option[27].

The martingale property of expected prices in sequential auctions is not the only change that result from considering more general demand functions. To start with, we need reconsidering the revenue equivalence theory (see Maskin and Riley (1989)). Thus, assume that a buyer has now an inverse demand function given by $h(q,v_i)$, where q represents the number of units of the good and v_i parameterizes the buyer's preferences. That is, $h(q,v_i)$ is a buyer's marginal willingness to pay for the q^{th} unit when her preferences are defined by the parameter v_i. We keep the assumptions about the stochastic process generating the v_i's. Also, we assume that $h(q,v_i)$ is decreasing in q, and it is increasing and differentiable in v_i. Once again, a trading mechanism could be described by the vector $\{\rho_i(v),\chi_i(v)\}_{i=1,2,\ldots,N}$, where now $\chi_i(v)$ represents the number of units that buyer i obtains if v is the realization

[27] In Burguet and Sákovics (1997a) we show that prices do decline if some buyer has flat demand for more than one unit, even though in this case the dissuasive effect of higher bids does not bite. Indeed, in the case analyzed, the "buyer's option" works as an endogenous uncertainty about the number of objects for sale. We also show that any uncertainty on this, endogenous or exogenous, will result in an expected declining pattern of realized prices even under risk neutrality.

of the taste parameters[28]. Then, similarly to Section 3.4, we can define the function $R_i(z,v_i)$ as the gross surplus that i expects if her type is v_i and she behaves as if it was z. That is,

$$R_i(z,v_i) = E_{v_{-i}}\left[\int_0^{\chi(v_{-i},z)} h(q,v_i)dq\right]. \qquad (40)$$

Also, define $p_i(v_i)$ exactly as in Section 2. Then, the trading mechanism satisfies incentive compatibility if

$$v_i = \arg max_z \; R_i(z,v_i) - p_i(z), \qquad (41)$$

whose first order condition is

$$R_{i(1)}(v_i,v_i) - p_i'(v_i) = 0, \qquad (42)$$

using again the notation for partial derivatives discussed in the previous section. Notice that, here too, the allocation of the objects (plus N initial conditions) determines the payments buyers expect (and then seller's revenues). Also, we can obtain the expression for the informational rents in terms of allocation of the objects. Indeed, integrating in (42), taking into account that

$$\frac{dR_i(v_i,v_i)}{dv_i} = R_{i(1)}(v_i,v_i) + R_{i(2)}(v_i,v_i) \qquad (43)$$

we obtain the net consumer's surplus for buyers:

$$R_i(v_i,v_i) - p_i(v_i) = R_i(0,0) - p_i(0) + \int_0^{v_i} R_{i(2)}(x,x)dx, \qquad (44)$$

and then substituting (44) in the seller's problem, which is still represented by (9), we obtain:[29]

$$Max \sum_i E_{v_i}\left[R_i(v_i,v_i) - R_{i(2)}(v_i,v_i)\frac{1-F(v_i)}{f(v_i)}\right] \qquad (45)$$

Notice that letting $h(q,v_i) = v_i$ for $q = 1$ and $h(q,v_i) = 0$ otherwise (and then $R_i(z,v_i) = x_i(z)v_i$ as defined in Section 2), we recover the unit demand case.

The conclusions from (45) are quite different from the ones obtained in the unit demand case. Indeed, we could generalize the definition of a uniform price auction as that in which each bidder can submit up to K bids for units of the object, and the K units are

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[28] For simplicity, we only consider deterministic mechanims, that is, we assume that after all buyers have made their moves the number of units each receives is deterministic.

[29] Branco (1996) generalizes this to the non-private values case, where h() depends on the whole vector v.

awarded to the K highest bids at a price equal to the K+1th highest. That is, one buyer could win more than one unit. Similarly, the general definition of the discriminatory auction would coincide with the uniform price auction except that all bids that are accepted are paid. Both institutions have been used to auction securities and public debt. It now happens that, in general, these two auctions do not solve (45), even when complemented with some reserve price. In general, the optimal auction is not a simple mechanism. It can be described as a non linear price rule with rationing: the seller fixes a payments function T(q), and then each buyer demands a quantity k_i given this non linear pricing rule. If total demand does not exceed K, then all demands are satisfied. Otherwise the demands are rationed according to some method specified in advance (Maskin y Riley [1989]). In the special case that demands are linear, that is, $h(q,v_i) = v_i - a\,q$, with a > 0, and for the regular case, this rationing is a proportional one: If $\sum_j k_j \leq K$, then each buyer receives her "bid". Otherwise i receives $k_i - \frac{\sum_j k_j - K}{N}$, (payments are, of course, computed according to (44)).

As we have already mentioned, neither uniform price or discriminatory auctions are optimal in general. Moreover, they are neither equivalent (in terms of seller's revenues) or, more surprisingly, efficient. Indeed, for the unit demand case efficiency was never a real issue: any of the most common auctions could guarantee it. Now, however, even efficiency will need special mechanisms. To illustrate this point in the simplest case, assume K=2. Consider a discriminatory auction. A bid can be thought of as a demand function, thus stating how much the buyer is willing to pay for her first unit and how much for the second. Now, bidding a demand function equal to the true $h(q,v_i)$ cannot be an equilibrium, since that would mean zero rents (surplus) for the bidder. Thus, as in the one unit case, bidders will "shade" their bids. However, this shading will be different for the first unit than for the second one. Indeed, the probability that a buyer obtains her first unit (i.e., the probability that her highest unit bid is among the highest 2 unit bids of all buyers) is higher than the probability of obtaining the second unit. Therefore, the trade-off (rents if winning versus probability of winning) is different for the two units and then the buyer will "shade" her two bids by different amounts (usually more for the first unit -her highest bid- than her bid for the second unit). But this implies that a buyer's bid for a second unit may be lower than some other buyer's highest bid even if the former values that second unit more than the latter values her first unit. Thus, inefficiencies will arise in a discriminatory auction with positive probability.

Perhaps more surprising than this result is the fact that similar inefficiencies arise when the seller uses a uniform price auction[30]. Again, submitting the true demand function as a "bid" (o set of two unit bids) is not the best response against virtually any rival strategy. Indeed, now there is a positive probability that the bid for the second unit (the buyer's second highest bid) is the third highest of all buyers' bids, that is, the one setting the price. Therefore, a bidder has an incentive to lower the bid for the second unit in order to decrease (with some probability) the price paid for her first unit, an incentive absent in her highest, first bid and very similar to the incentive a monopolist has to increase the price above marginal cost. Thus, again, a buyer can obtain her first unit even if she values it less than some other buyer values her second unit. In both cases, the inefficiency takes the form of a tendency towards disseminating the units across buyers more than what relative willingness to pay would indicate. Which of the two auctions is more efficient or gives more revenue to the seller depends on the particular parameters of the situation (see Ausubel and Cramton (1998)).

There are, however, relatively simple auctions that can guarantee efficiency in this context. In fact, Vickrey (1961) had already proposed an efficient auction, one that correctly generalized his second price auction to the multi-unit case. This auction has everybody submitting a demand function, as the uniform price or the discriminatory auctions. However, buyer i pays for her k^{th} unit received a price equal to the k^{th} highest rival bid rejected. More recently, Ausubel (1997) has proposed what would be the oral, ascending version of it in as much as the English auction can be considered the oral, ascending version of the second price auction. In Ausubel's auction, bidders answer to the seller's (continuous and increasing) price announcements by declaring how many units they are willing to buy at that price. The process continues (with the restriction that a buyer

[30] Actually, it should not be so surprising, since Vickrey (1961) had already underlined this fact. However, even thirty years later, economists as reputed as Milton Friedman or Merton Miller still thought (and made it public) that uniform price auctions had truth as a dominant strategy (see Ausubel and Cramton (1998)).

cannot increase her demand as the price increases) until the first price at which total demand is equal (or lower) than K. Demands are then satisfied. How much does a buyer pay for a unit? The answer is the price at which she "clinched" that unit. To illustrate the workings of the auction, let us come back to the K=2 example, and assume N=2. The price starts at zero and increases continually until one bidder, say bidder 1, reduces demand from 2 to 1 units. At that point bidder 2 has already clinched one unit. That is, whatever happens afterwards she will receive one unit: bidder 1 has decreased demand to 1 and there are 2 units in total. Then that is the price bidder 2 will pay for that unit. The process continues until a price at which one of the two bidders reduces her demand in one unit, which is the price that the other bidder will pay for that unit. It is a simple exercise to prove that bidding "truthfully", that is, asking at each price for a number of units equal to the true demand at that price, is a dominant strategy, as in the Vickrey auction. Then, again, all units will be allocated to those bidders who value them most (and seller's revenues are the same in both, oral and sealed bid, auctions, according to (42)).

So far we have been talking about "downward sloping" demands. In the last years, however, a new interest has arisen on what one could term as "upward sloping demand" auctions. That is, auctions of multiple objects with the property that buyers' valuation for two units is more than twice their valuation for one unit. The interest in these cases, referred to as auctions with synergies, increasing returns, or superadditive values, has to do with the much publicized and already mentioned auction of spectrum rights in the USA. This was conducted using a simultaneous, ascending auction[31]. What was new about that auctioning of a number of different licenses covering different territories and different bands of the radio spectrum was the fact that two adjacent licenses, both geographically and in terms of frequency, were worth for at least some bidders more than the addition of the two individual licenses. Indeed, they would allow to exploit economies of scale in territory covered by the new personal communication systems they could carry, and in the amount of spectrum that could be gained if frequencies did not have to be separated. The problem of designing optimal (or satisfactory) auction mechanisms for this type of goods is still very

[31] See, for instance, McAfee and McMillan (1996) and Cramton (1995) for a description and early analysis of these auctions.

much open. After a first wave of excitement about the simultaneous ascending design used by the FCC in the US, some have started questioning the efficiency (the main goal of the FCC) of such a design (see, for instance, Ausubel and Cramtom (1998)). In fact, there are very few theoretical analysis of auctions with synergies. One exception is the work by Krishna and Rosenthal (1996), where they analyze equilibrium bidding behavior in a simultaneous, second price auction of two goods in presence of "global bidders", those with demand for more than one object and synergies, and "local bidders", those interested only in one of the objects for sale.[32] The way the authors model synergies is by assuming that a global bidder has some (private and independent) valuation v for one good, but $2v+\alpha$ for both objects together. Apart from being complex (and discontinuous in general), the global bidders' behavior has some interesting properties. First, for very low values of v the global bidders bid less than their per unit value (i.e., $v+\alpha/2$) for each object. The reason is that the chances of obtaining both objects are very small, and then they bid more as if expecting to gain only one object (which they value individually in v). However, for very high realizations of v global bidders bid for each object more than their per unit valuation. Indeed, in that case it is very likely that they win both objects, but their marginal valuation for each of the objects conditional on having won the other is $v+\alpha$. Thus, global bidders have both regions of "overbidding" and regions of "underbidding". Krishna and Rosenthal also show that an increase in the number of bidders does not always lead to an increase in competition. In fact, an increase in global bidders always leads to less aggressive bidding by the global bidders. In terms of revenue comparison, they show that the ranking of sequential and simultaneous auctions is ambiguous.[33]

Considering multiple unit auctions opens yet another important question which was absent in the single unit auction: the possibility that a seller has different valuations for different units too. Indeed, we have been assuming that the seller valued all units equally, which we normalized to zero. This needs not be the case (governments may decide to sell more public debt if the "offers" to buy include lower prices or bids, for instance). That is, the seller's "supply" may have positive "price elasticity". Moreover, some recently created

[32] See Rosenthal and Wang (1996) for a similar analysis in a common value setting.
[33] See Branco (1997) for an analysis of sequential auctions with synergies.

auction markets have these elasticities as an important component to be taken into account. Changing the role of buyers and sellers, examples of this are some spot markets for electricity. Indeed, regulatory reform has introduced in some countries not only auction mechanisms to allocate power generation needed to satisfy demand, as in the British case, but also bidding on the part of the demand side (large consumers, electric utilities, ...), as in California or Spain.

The study of such auction markets can be approached by first slightly generalizing our analysis of multiple object auctions.[34] Indeed, instead of a fixed amount K, assume now that the seller announces an increasing supply schedule S(p). On the other hand, assume that buyers have flat demand schedules, that is, $h(q,v_i) = q\ v_i$. This is the model analyzed by Hansen (1988), which, exchanging the roles of buyers and sellers, can be interpreted as a model of bidding for the right to satisfy a downward sloping demand (in our case, to buy from an increasing supply S(p)), or an auction with endogenous quantity. Here we could restrict both uniform price and discriminatory auction rules so that buyers could only submit flat demand schedules, that is, offer a price. Then, in a uniform price auction the winner would be the one offering the highest bid or price p_i but would pay the second highest bid p_j per unit and obtain $S(p_j)$ units of the good. In the discriminatory auction, the highest bidder would win again, but would buy an amount equal to $S(p_i)$ and pay her bid, p_i, per unit.

It is not difficult to see that bidding a price equal to the true valuation v_i is once again a dominant strategy for this version of the uniform price auction. For the version of the discriminatory auction and symmetric bidders, one could characterize a monotone symmetric equilibrium bidding function $p(v_i)$ in a way similar to that used in Section (2) when we analyzed first price auctions. That is, one would have to have

$$v_i = \arg max_z [v_i - p(z)]F(z)^{N-1}S(p(z)) \tag{46}$$

with first order conditions that could be written as

$$p'(v_i) = \frac{[v_i - p(v_i)](N-1)f(v_i)}{F(v_i)\left(1 - [v_i - p(v_i)]^{S'}\!/_{S}\right)} \tag{47}$$

Now, again using the Lemma in Section 3, we will conclude that the expected price in the discriminatory auction is higher in the endogenous quantity case than in the fixed quantity

[34] The analysis will be deepened with the study of double auctions in the next section.

analyzed in Section 2. Indeed, a zero valuation buyer would have to bid zero in equilibrium, as in the fixed quantity case. Also, if at any point the bidding function in the fixed quantity case $b(v_i)$ were equal to $p(v_i)$, then the numerator of (47) and (3) would be the same, whereas the denominator for (47) would be lower (as far as S' is positive, as we are assuming), and therefore p' would be higher than b' at that point. According to our Lemma, this allows us to conclude that $p(v_i) > b(v_i)$ for all $v_i > 0$. But this also implies that in our endogenous quantity case the expected price (and the quantity sold) is higher in the discriminatory auction than in the uniform price one: in the latter it is the same as in the second price auction, which in turn is equal to the expected price in the first price (fixed quantity) auction, this one lower, as we have just seen, than the expected price in the discriminatory auction for endogenous quantities.

The intuition behind this result is simple. Indeed, in the discriminatory auction when one wins all or nothing, there is a incentive that works against shading the bid (which is represented by the extra term in the denominator of (47)): shading the bid makes the price fall, but the amount obtained is also reduced. However, in the uniform price auction in which a bidder wins all or nothing this incentive is non existent, since the price, and therefore the quantity, is not directly affected by the winner's bid. Moreover, the fact that quantities sold are increased (in expected value) also leads to efficiency gains, since the outcome of (any) auction would have the characteristics of oligopoly output, that is, will be lower than that which equates marginal willingness to pay with marginal cost.[35] Hansen shows that in general this efficiency gain will make both buyers and seller better off.

The analysis above assumes constant marginal willingness to pay, that is, flat demand curves. Relaxing this assumption seems important when one is interested in studying cases as the spot markets for electricity mentioned above (again, exchanging the role of buyers and sellers). Indeed, there a plant is unable, in general, to supply the whole market. That is, they are capacity constrained. Moreover, usually a bidder owns more than one plant, perhaps with different costs. Thus, a better model for this case would assume buyers (sellers, in power generation) to have decreasing, may be piecewise constant,

[35] We are using a concavity argument here, since the output in a uniform price auction would be more "stochastic" than the one from the discriminatory auction.

marginal willingness to pay. That is the model analyzed by de Otto (1997) using the uniform price auction except that the price is the bid by the marginal plant that does produce (in general, not at full capacity), not the first one to be excluded. Thus, for inframarginal bidders, the situation is similar to the one faced by Hansen's bidders under the uniform price rule, whereas for the marginal bidder the situation is more like Hansen's bidders under discriminatory auction. Notice that capacity constraints absent, we would be dealing with this discriminatory auction. Then firms are more aggressive (ask for lower prices or, if it is the buyers' side that bids, offer higher prices) because they are capacity constraint. The final effect on the expected price, however, is ambiguous (now it is not necessarily the most efficient plant, o highest valuation buyer who sets the price). With respect to ownership structure, now bidders have an incentive to manipulate the bids of the less efficient units in order to affect the price for the inframarginal ones, an incentive very similar to the ones described by Ausubel and Cramton (1998).

We are now very close to the analysis of auctions as mechanisms for a real price formation in markets. The only thing that is missing in this regard is the presence of strategic agents in both sides of the market that take the mechanism as given . Indeed, up to now we have only considered one strategic side of the market (buyers) while the other side's role, if any, was to define the rules of the game. In the next section we take the last step forward to enter into the analysis of auction games as markets.

5.- Auctions and markets

Some organized markets, like stock markets, are actually auctions in which both parties to the trade bid for exchange. This special type of auction is known as double auction.[36] One can distinguish oral double auctions from "sealed bid" double auctions, some times referred to as clearing houses or call markets. In the first, agents meet, buyers announce bids and sellers asks at any moment in time, and trade occurs whenever a seller accepts a bid or a buyer accepts an ask. In the "sealed bid" double auction, buyers and sellers simultaneously submit their bids and asks (perhaps for multiple units), then an "auctioneer" draws a demand

[36] This mechanism, for the case of one buyer and one seller, has received extensive attention as a model of bilateral bargaining. See Myerson and Satterthwaite (1983) for an early study of this case.

function with the aggregation of the bids and a supply function with the aggregation of asks. A market clearing price for such supply and demand schedules is determined and buyers who set bids above this price receive their corresponding units from sellers who set asks below that price (see Figure 1). Given that in general for indivisible units there will be a continuum (an interval) of market clearing prices, $[\underline{p}, \bar{p}]$, one usually refers to an auction picking a convex combination $k\bar{p} + (1-k)\underline{p}$ as the k-double auction.

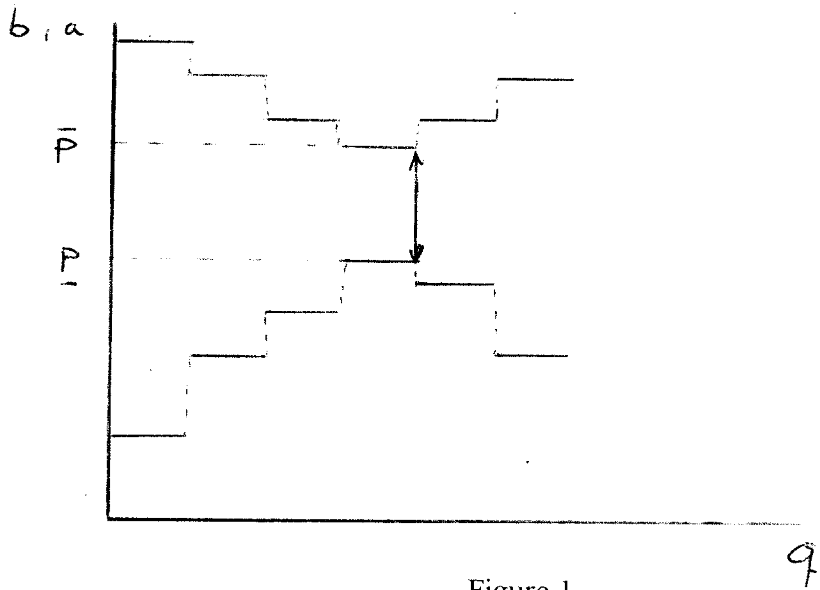

Figure 1

Double Auctions

The analysis of oral, continuous time double auctions is almost hopelessly difficult. Most of the analysis of this institution has taken the form of controlled, laboratory experiments (see Friedman and Rust (1993) for a collection of those studies)[37]. The analysis of the k-double auction is more tractable, under unit individual supplies and demands. Here one typically assumes that N buyers as characterized in Section 2 and M (= N, usually) sellers characterized by some i.i.d. valuation or reservation value c_j with some known c.d.f. and density function G and g, simultaneously decide their bids and asks, respectively. Even though no general results on existence are available, at least the first order conditions for symmetric equilibrium are not hard to obtain. An additional difficulty with double auctions,

[37] Easley and Leyard (1993) study this institution from a non-Bayesian, non-game theoretic angle, postulating a set of plausible behavioral rules agents would use.

though, is the inherent multiplicity of (even symmetric, monotone) equilibria. The origin of this in the fact that initial conditions for what is now a differential equation system, that is, the first order conditions for the seller's and the buyer's strategies, are somewhat endogenous. Indeed, knowing what is the lowest possible ask α a seller sets in equilibrium (for monotone equilibria, this is the ask of a seller with lowest possible valuation or reservation value), we have initial conditions for the buyer's bidding strategy: a buyer with valuation α bids exactly α. Similarly, knowing the maximum possible bid β a buyer may make in equilibrium (the bid of a buyer with highest possible valuation), we have initial conditions for the seller's strategy: a seller with valuation or reservation value equal to β sets an ask of β. The problem is that what is the highest equilibrium bid depends on the initial conditions for the bidding strategy, which in turn we have seen depends on the lowest equilibrium ask, this one depending on the initial condition of the ask strategy. This circularity explains the generic multiplicity of equilibria (see Satterthwaite and Williams (1993)).

Even without closed forms for equilibrium strategies, there are interesting things that can be said about k-double auctions. In particular, (see Rustichini, Satterthwaite, and Williams (1994)) both misrepresentation of valuations and the inefficiency arising from it (buyers bid lower than their valuation and sellers' asks are above their reservation value, and then some trades that should take place don't) converge to zero as the number of agents converges to infinity. Moreover, the convergence to efficiency is fast, at least for the uniform distributions case, in the sense that it attains the convergence speed of a theoretically optimal mechanism, computed in Gresik and Satterthwaite (1989). Thus, a double auction mechanism is a model of Walrasian price formation at least for a large number of traders, the case the Walrasian model studies, anyway.

There is, however, yet a step to be taken if we want to get close to the idea of a market as a decentralized institution for interaction of buyers and sellers. This is to dispense with the "center" needed in a double auction mechanism. Indeed, in a ("sealed-bid") double auction "someone" receives bids, compares them, and sets prices and trades according to some previously specified rule. Nothing of this is present in markets other than organized ones. Thus, we analyze now models of "auction competition", as idealized representations of markets where sellers and buyers interact in a decentralized way.

We could begin the journey with a paper by Bulow and Roberts (1989) that compares the problem faced by the seller when designing an optimal auction mechanism with that faced by a monopolist when setting optimal prices for separated markets.[38] They show that these two problems are equivalent. Indeed, let us come back to the problem studied in Section 2, and consider each buyer i as a market. If offered a price P_i, this buyer would be willing to take the good with probability $1 - F_i(P_i)$ (we don't need to assume buyers symmetry for this analysis). Therefore, considering probability as quantity, which for a risk neutral agent is the same, this would be the "demand function" $D_i(P_i)$ from this buyer i when considered as a market. The optimal policy of a monopolist with zero marginal cost, the seller's valuation, would be to set a price P_i for market i that solves:

$$P_i = \arg max_P D_i(P)P = \arg max_P (1 - F_i(P))P. \qquad (48)$$

That is, P_i would be defined by $J_i(P_i) = 0$, the price that equals marginal revenue with marginal cost. Which is exactly the "personalized" reserve price that defines the optimal auction.[39]

This parallel is surprising in that, even accepting probability as demand, the reserve price is not the price a seller obtains in the auction, in general. The price is determined by the interaction of buyers. That is, buyers are not "separated" markets.

If we go to the other extreme of the spectrum of market structures, we could obtain similar parallels. Indeed, consider competition by sellers in the following form: sellers announce their "auction mechanisms", then buyers decide which (one) seller to visit, and then auctions are carried out. Although it is not necessary, assume that sellers' mechanisms are all of the second price type. That is, they compete for buyers by announcing (may be different) reserve prices. McAfee (1993), Peters and Severinov (1997), and Peters (1997) show that when the number of sellers is very large, the reserve price in this "market" will be set to marginal cost[40], that is, equal to the valuation of the seller (zero, in our examples).

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[38] See Wilson (1977) for an earlier, less explicit paralel between market price and auction price.

[39] Assigning the object according to virtual valuations is, therefore, as selling to the market with highest marginal revenue.

[40] This is not a Walrasian result, though, since in these models the probability of more than one buyer showing up at the same auction is positive (then having a trade at a price above marginal cost), and so is the probability that a seller gets no buyer then having Walrasian trades that are not realized.

Thus, one can wonder what is there in reserve prices that seems to make them so close to ordinary prices. In particular, one could conjecture that indeed the above type of auction competition is a model of price formation for "any" market structure.

Thus, consider M sellers, each offering a unit of a homogeneous good, who compete for a clientele of N buyers defined as in Section 2. Again, assume that competition takes the form of announcing reserve prices for their second price auctions. After observing these announcements, each buyer decides what auction (if any) to take part in. Is this model equivalent to any of the oligopoly models? Is the reserve price equivalent to the price in any of these models with only considering the probability of a buyer attending the seller's auction and being willing to take the reserve price as the (residual) demand of a market?

The answer is in the negative (Burguet and Sákovics [1997b]). To understand why, let us come back to the result in Bulow and Roberts (1989), and think of the sellers as using second price auctions. In the "monopoly" case, the reserve price has an effect on the price received by the seller if it happens that only one buyer has a valuation above the reserve price. In any other case, the price would be equal to the second highest valuation, and therefore an infinitesimal change in the reserve price does not affect the seller's revenues. But when only one bidder shows up, the reserve price is actually the price.

However, whenever the reserve price has any effect on the seller's revenues even conditioning on more than one buyer showing up, the parallel disappears. That is what happens in the "oligopoly" model we are considering here. Indeed, take the case in which M=2. If sellers announce reserve prices P_1 and P_2 ($>P_1$) the decision of what of the two auctions to attend will be defined by some cutoff level $w(P_1,P_2)$, below which a buyer only considers attending auction 1 and above which buyers randomize (with equal probability) among the two auctions. The important fact is that $w(P_1,P_2) > P_2$ and is increasing in P_2. Indeed, a buyer with valuation just above P_2 will still prefer to attend auction 1, where the probability of winning is positive and the expected price in that case is strictly below her valuation. But at some point buyers would have to attend auction 2 with some probability in equilibrium (unless P_2 is very high and close to 1), since otherwise the price at auction 2 would be P_2 for any bidder that deviates and attends that auction, where she can win with probability 1. On the other hand, the higher P_2 the higher the cutoff level above which a buyer may consider attending that auction 2. Similarly, $w(P_1,P_2)$ is decreasing in P_1. But this means that the reserve price will have an effect on the seller's revenues even conditioning on more than one buyer showing up in her auction: the distribution of those buyers' valuations will be different for different reserve prices. That breaks the parallel between reserve prices and prices too.[41]

One could consider any model like the ones discussed so far as still not fully satisfactory model of price formation in decentralized markets. It is true that we are already talking about decentralized competition in a sense (there is no central organizer in these "markets"), but each auction becomes a "separate" market before the price is formed. That is, by the time bidders bid, they are already committed to only that auction. A more attractive auction-based model of price formation for decentralized markets would have bidders submitting bids to possibly all auctions, in a way in which every party moves before the price(s) and trade(s) are defined. Burguet and Sákovics (1998) study one such model. There sellers (with possibly different "costs" or valuations) announce their reserve prices. Then buyers decide what offer (bid) to make to each seller. Sellers receive all offers and then decide which one to accept. A buyer finally chooses what trade to execute among all of her accepted offers. If there are still sellers who did not get to sell but have standing offers from buyers who did not get to buy, then these sellers decide again what of those standing offers to accept this time, etc. We show that, as the number of agents grow, the outcome approaches the Walrasian outcome of a market with demand function obtained from aggregation of buyers' willingness to pay and supply obtained by aggregating the sellers' (unit) costs or valuations. In the limit, there is no variance in prices[42] and the common price is the competitive price. That is, we present an auction-based, decentralized model of price formation of a competitive market where prices are formed by the interaction of a large number of buyers and sellers.

This is not the only possible way to formalized price formation in presence of incomplete information. However, it illustrates the potential that a rigorous analysis of

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[41] In Burguet and Sákovics (1997b) we show that the equilibrium (usually in mixed strategies) reserve prices will be different from zero with probability 1.

[42] Note that in the (sealed bid) double auction mechanism price dispersion was assumed away: for any actions taken by the traders, the mechanism selected a unique price at which all transactions would take place.

auction-like models has as a way to understand the way prices are indeed formed in markets. This is probably the farthest reaching horizon for auction theory in the coming years.

6.- Some closing remarks

We have come to the end of a tour, one among many possible ones, to the theory of auctions. The emphasis has been on understanding how agents behave strategically in trading situations, and on the effect of the rules of trade on the outcomes. In this sense, the last part of our walk had to be reserved to what auction theory has to say about price formation in markets. We have taken both positive and normative approaches, looking at the design of optimal (or satisfactory) institutions, and looking at the workings of well known ones too. Many things had to be left aside. To cite only a few, we have not touch on the issue of collusion and its effects on the workings of auctions. Indeed, we have been always assuming that each buyer acted independently of others. The issue of buyers' collusion, however, is an important one especially when the same set of buyers interact often (contract bidding, etc.). See McAfee and McMillan (1992) for one study of the effects of collusion.

Multidimensional auctions is another important topic we have not analyzed. This is of particular relevance in procurement or public works contracts, where the "goods" that several sellers may offer differ in quality or other non-price dimensions. The auction design has to weigh these different dimensions and provide the correct incentives for the mix, and at the same time limit informational rents (see Che (1993) and Branco (1997)). And the list of other topics could continue with the inclusion of income effects, for instance due to financial constraints suffered by the bidders (see Che and Gale (1994)), or the analysis of auctions for heterogeneous (from the buyers' viewpoint) goods (see Bernhardt and Scoones (1994)), etc. But rather than keep going on with references to all that has been analyzed and discussed in the literature, the above pages try to synthesized the basic tools and themes of a theory that today, I believe, represents the most solid grounds on which to construct a theory of market and institution design in presence of incomplete information.

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