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Abstract

We model the incentive problems derived from the presence of a large inherited debt. It has been argued that the distortions caused by the inherited debt are such that a partial forgiveness on the part of the creditors might be Pareto improving. We study this feature in a dynamic context. The bank faces the problem of designing the intertemporal forgiveness scheme.

In case it wants to partially give up its rights over the country's product so as to incentive its effort, it must decide the way to do it through time. We allow it to fix a vector of upper bounds over the amounts that will be repaid in each period. We find it is optimal to establish the creditor's rights as a one period maturity debt, i.e., an amount from which repayments are subtracted and which grows at a rate equal to the inverse of the common discount factor. The initial debt involves a partial forgiveness, as in the one period model, and is such that, if a bad state of nature occurs, it will grow beyond its optimal level so that a further reduction will be called for.

Thus there is an initial forgiveness and a further future forgiveness in case the country gets a low product and makes a small repayment. This feature is due to credibility motives: the bank would like to punish harder a bad product outcome, but if such a situation occurs, it will not find in its interest to do so.

We find the debt reduction argument is reinforced when placed in a dynamic context. It is possible that the bank does not want to forgive debt if the relationship is one shot, but it does if it is more lasting.

The optimal repayment scheme induces a country's effort above the one period equilibrium level. There are situations in which the debt overhang could even cause over effort.

We make comparative statics as the interest rate varies and find that, starting from a high enough debt level, an increase in the interest rate benefits the country and induces a greater equilibrium effort.

0. INTRODUCTION

Between 1982 and 1985 many less developed countries with big foreign debts did not meet their obligations as originally agreed. This was the beginning of the so-called external debt crisis. Several aspects of it have been studied. One of them is the incentive distortions it causes.

It has been argued that the presence of a large inherited debt distorts the actions that a country undertakes. The reforms to stabilize the economy and foster growth require a sacrifice. In the presence of a large inherited debt, at least part of the benefits it produces will accrue to the creditors because the service of the debt will then be higher. Likewise, a potential investor will anticipate that the debt will act as a tax on his (her) investment. Both the investment and the reforms will look less attractive to those who decide the extent to which they are undertaken. It has been argued that these distortions could be so high that a partial debt forgiveness would be Pareto improving. The level of debt that the creditor finds it convenient to establish comes out of a trade-off between giving incentives to the country to undertake costly actions and getting benefits from these actions. Most models studying this problem are static, in that both the debtor and the creditor act only once. The bank chooses a level of debt and the country chooses a level of effort (or investment) and repays only once. Nevertheless, the relationship between a bank and a country is more lasting. It repeats itself over time and contains, therefore, features of intertemporal nature that cannot be captured by models in which each agent acts only once. We model the bank-country relationship as one which repeats itself over time. This is the contribution made by the models we build. We study the incentive problem mentioned above in a dynamic context.

In a static model, the country knows the level of debt chosen by the bank and, in deciding its action, computes the fraction of the returns that it will be permitted to keep. In a more lasting relationship, if it is true that these incentive problems exist, the country will anticipate, when deciding its actions, that the bank might find it convenient to partially forgive debt for the same reasons it has done it before. As for the bank, if it has to forgive debt, it not only has to decide how much to forgive, but how to distribute this forgiveness through time. These are important features which do not appear in static models.

The rest of this paper is organized as follows.

In section 1 we expose the basic argument for debt reduction. We build a model in the spirit of Krugman (1988) and Sachs (1988 a, 1988 b), who argue that debt reduction could conceivable benefit not only the debtor, but the creditor as well. In the model we present there exists a large inherited debt. The amount received by the bank depends on whether the country's product is high or low. The country affects, through its effort, the probability of obtaining a high product instead of a low one. On the other hand, effort is costly to the country. The initial debt is such that if it were not reduced it would allow the bank to confiscate in all cases the entire country's product. In equilibrium there is debt reduction and effort is, although greater than without debt reduction, still smaller than the socially optimal level.

In section 2 we build a dynamic model. To simplify we study the two-period case. The intertemporal nature of the forgiveness scheme the bank chooses depends crucially on the commitment capacity it has. We allow it to establish at the beginning of the game a vector of upper bounds to the amounts of product it can take from the country in each of the two periods which the relationship lasts. The bound for the first period is just a real number. The bound for the second period is in fact a function. We allow it to commit itself for the second period to a bound contingent on the amount received in the first period. This makes sense under the (realistic) assumption that the amounts received by the bank are verifiable by outside observers. At the beginning of the second period, the bound that comes out of the first period payment is legally binding. But it is just an *upper* bound. We assume the bank can give an additional forgiveness if it wishes to do so.

In section 3 we analyze the bank's strategy in equilibrium. There is more than one optimal way to forgive, but the optimal contract satisfies the following condition. It does not leave a level of debt that causes disincentive problems tomorrow if it can be exchanged for today's product. Among the optimal ways to forgive debt there is one which establishes the creditor's rights as a debt with one period maturity, that is, an amount from which payments are subtracted and whose unpaid part grows at a rate equal to the inverse of the common discount factor. Under this scope, the bank's strategy comprises an initial debt reduction and a further reduction contingent on the first period outcome. The additional forgiveness is made in case the first period product is low and the country makes a small repayment. This feature is explained by credibility problems: the bank would like to penalize the country more severely in case a low product is obtained, but if this happens it will not find in its interest to do so.

In section 4 we analyze the country's effort. We find that the country's strategy specifies, when it meets the optimal repayment scheme, an initial level of effort which is higher than the one period equilibrium level.

In section 5 we study the following problem. During the initial years of the debt crisis, the policy to face it was one of postponing the payments that were due in the hope that they would be eventually met. It was a temporary relief policy. With this behavior as a motivation, in this section we restrict the bank to start the second period with a large debt. We find that this restriction harms both the bank and the country.

In section 6 we study what happens to the partial forgiving of debt argument when we put it in a dynamic context. We find that it is strengthened in the following sense. It may happen that the bank does not find in its interest to forgive debt from a static viewpoint (if it were only one period), but it does from a dynamic perspective (knowing they have a more lasting relationship).

Up to the previous section we assume that the initial rights of the bank are such that it can withdraw the entire product from the country. In section 7 we study what happens if the initial situation is not so extreme. The answer depends crucially on the productivity of the country's effort. For low productivity values, the answer is the intuitive one: the higher the initial rights of the bank, the lower the level of effort the equilibrium will exhibit. For higher productivity values, the equilibrium effort is not monotone in the burden of the initial debt. It can even happen that for some levels of debt, the equilibrium exhibits an initial effort level higher than the socially optimal one.

In section 8 we come back to the large inherited debt assumption and ask ourselves what happens if the country and the bank have different discount factors. This allows us to make comparative statics for the interest rate. We find that, starting from a sufficiently large initial debt, an increase in the interest rate benefits the country and induces a greater initial effort. The first of these characteristics appears, in a different context, in Bullow and Rogoff (1989)

Finally, in section 9, we extend some of our results to models with arbitrary (finite) number of periods.

Two recent papers related to ours are Borensztein (1990) and Calvo and

Kaminsky (1991).

Borensztein (1990) also studies the debt overhang argument in a dynamic context. To explain the difference between his work and ours, let us first remember the next fact, also mentioned by this author. When the problem of a large inherited debt is modeled through investment there are two ways in which this second variable is affected. On the one hand the investment decision is distorted because the marginal returns will be shared with creditors (incentive problems) and, on the other hand, the lack of foreign credit causes the internal interest rate to be higher than the international one: even though investors received the entire proceeds of their investment, if the country cannot receive foreign credit, projects which are valuable at the international interest rate might not be undertaken (rationing problem). We will study only the former problems. We will model the debt overhang a la Krugman (1988), that is, having as the country's decision variable effort and not investment, so that we can concentrate on the pure incentive problems. Borensztein (1990) builds models with investment and compares the effect of credit rationing with that of incentives. However, the intertemporal nature of the repayment scheme is not a decision of the bank, but a fixed proportion of the country's output through all the periods. This is the main difference of their models with ours. We endogeneize the intertemporal structure of the repayment scheme. Borensztein (1990) is interested in different problems. He compares the relative influence of the two mentioned effects upon investment for different values of the parameters of the model.

Calvo and Kaminsky (1991) use the optimal contract approach to study debt relief, both in the static and the dynamic case. In their models the country's product does not depend on any variable under the country's control. They are interested in the optimal way to share risk when the country is risk-averse and the bank risk-neutral.[1] In our work it is essential the influence of the country's actions upon its product. On the other hand, the risk sharing problem does not appear.

A final word of caution. All our models belong to the class that starts with a large inherited debt and, assuming there is a mechanism which causes that countries make repayments to banks, studies several features arising from this situation. We assume that a mechanism exist which guarantees that (at least) part of the country's output is transferred to the creditor. Why this is so - why countries repay debt, which leads us to ask why banks lend to countries in the first place- is a question that has given rise to many papers[2]. In a financial transaction there is a difference between the time at which the lender gives the money and the one at which the borrower must return it. In domestic credit markets, the force of the law guarantees that the second agent fulfills his obligation. Furthermore, it can often be deprived of a collateral in case of default. In international markets, there do not exist courts to which the lender can ask for his rights to be respected. If we consider as collateral the assets of the country out of its territory, we find it is very small and, in case of being able of translating more of them out of its territory, presumably they would loose their usefulness.

The idea that 'reputation' could act as a collateral was considered as a convincing answer during some time, i.e., if a country does not repay its debt, nobody will ever lend to it again and because of this it will prefer to repay. Bullow and Rogoff (1989) prove that, under very general conditions, the previous argument is not valid. Essentially, if a country deprived of the possibility of borrowing resources can, however, lend (save) in the international markets, then the reputation argument falls apart: the country would rather default and save instead of repay and keep its reputation. So, it is not possible to establish a reputation for repaying debts and the loans to less developed countries are possible only if creditors can affect countries' interests. Along these lines the most plausible candidates are the ability to disrupt international trade and (perhaps to a lower extent) to confiscate the country's assets abroad.

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[1] They assume that it is costly to verify the realization of the product (otherwise the solution is trivial) and restrict themselves to a special class of contracts: those in which the cost is incurred in if and only if the product is smaller than a certain level.

[2] A survey of many of them and, in general, of the literature up to 1986, can be found in Eaton, Gersovitz y Stiglitz (1986)

1. A STATIC MODEL

The Model

In this section we consider a static model in which, starting from a large inherited debt, a bank finds in its interest to reduce it. It is a model in the spirit of Krugman (1988) and Sachs (1988 b) who argue that the existing debt creates strong distortions in the incentives the country has to undertake actions which have a positive effect on its product but are costly.

We start from a situation in which a country owes a debt D to a bank. The country makes an effort e which, together with a random state of nature, determines a product x, measured in the same units as D. We assume that the support of the random variable x is the set

$\{ \underline{x}, \bar{x} \}$ with $0 < (\bar{x} - \underline{x}) < 1$ and its distribution is determined by

$$P(\bar{x}) = e \qquad (1)$$

That is, it can happen a high ($\bar{x}$) or a low ($\underline{x}$) product and the probability for the former is increasing in the country's effort (for simplicity we assume it is equal, so as to obtain closed form solutions). We assume $\bar{x} < D$, a situation that produces the incentive problems in their purest form.

Consider the following three-stage game. In the first stage, the bank reduces the debt [1] to a level $\bar{R} \in \mathbb{R}$. In the second stage the country chooses an effort level $e \in [0, 1]$. Finally, nature chooses a product x using the rule (1) and the bank receives the amount

$R = \min \{ \bar{R}, x \}$ (gun-boat assumption)[2]

that is, if the country does not repay $\bar{R}$, the bank can withdraw all the

[1] Note that we restrict the bank to fix repayment schemes in debt form. It is worth studying them because they are widely used. If effort is verifiable and effort-contingent repayment schemes are allowed then the bank can induce the country to choose the first best level of effort while obtaining the same utility it would obtain from exerting no effort. On the other hand, allowing for product-dependent contracts, when the product can take on only two values as is our case does not change the results.

2.This assumption is also made by Krugman (1988) and Froot (1989). A model in which the fraction of the product received by the bank is endogenous (arises from a bargaining process) is that of Bullow and Rogoff (1988).

product. Under this hypothesis we can study the incentive problems derived from the ability of the bank to deprive the country from part of its product in its purest form.

The time line for choices is as follows.

Figure 1.

bank fix $\bar{R}$	country chooses e	realization of x. bank receives $R = \min \{ x, \bar{R} \}$

Agents' preferences can be represented by the following VNM utility functions

country $H(x-R, e) = (x - R) - g(e)$

bank $G(R) = R$

We assume

$$g(e) = e^2/ 2 ,$$

unless otherwise stated, so as to obtain closed form solutions. That is, effort is costly to the country and the cost ($g(e)$) increases in an increasing way with effort.

Note that a risk-sharing problem will not appear here because both utility functions are affine in the received product.

In this game a strategy for the bank is an upper bound $\bar{R}$. A strategy for the country is a function that assigns an effort level e to each $\bar{R}$.

Equilibrium concept

The following two preliminary definitions will be helpful in defining the equilibrium concept.

Let $C(e, \bar{R})$ be the expected utility for the country when an effort level e and an upper bound $\bar{R}$ are chosen. Let $B(e(\bar{R}), \bar{R})$ be the expected utility for the bank when a function $e(\cdot)$ and an upper bound $\bar{R}$ are chosen.

The equilibrium concept we use is that of subgame perfection[3]. We define

[3] Note that the country's strategy must specify an effort for each upper bound.

it as follows.

An optimal self-enforcing contract (OSEC) is a pair of strategies $(\bar{R}^*, e^*(\cdot))$ such that

$$
\begin{array}{llll}
e^*(\bar{R}) & \in & \underset{e}{\operatorname{argmax}} & C(e, \bar{R}) \quad \forall \ \bar{R} \\
\bar{R}^* & \in & \underset{\bar{R}}{\operatorname{argmax}} & B(e^*(\bar{R}), \bar{R})
\end{array}
$$

Analysis

The following proposition tells us that if we allow the bank to make an arbitrary decision about the level of debt, it will choose an amount which is strictly smaller than the maximum conceivable product. The reason is that the country's incentive restriction poses a trade-off between amount of debt and probability of receiving this amount. The higher the debt, the higher the probability of not receiving it (and getting only $\underline{x}$). Notice also that the optimal choice for the bank induces an equilibrium effort smaller than the first best level.

Proposition 1

The only optimal self-enforcing contract is

$$
\bar{R}^* = \frac{\bar{x} + \underline{x}}{2} \equiv M < \bar{x} \qquad\qquad e^*(\bar{R}) = \begin{cases} \bar{x} - \underline{x} & si \ \bar{R} < \underline{x} \\ \bar{x} - \bar{R}, & si \ \underline{x} \leq \bar{R} \leq \bar{x} \\ 0 & si \ \bar{x} < \bar{R} \end{cases}
$$

which induces an equilibrium effort $e^*(\bar{R}^*) = (\bar{x} - \underline{x})/2$ smaller than the first best level equal to $\bar{x} - \underline{x}$

Proof

Since $R = \min \{ x, \bar{R} \}$, the expected utility for the country is

$$C(e, \bar{R}) = E\, H(\bar{x} - R, e) = \begin{cases} \underline{x} + e(\bar{x} - \underline{x}) - \bar{R} - e^2/2 & \text{if } \bar{R} < \underline{x} \\ e(\bar{x} - \bar{R}) - e^2/2 & \text{if } \underline{x} \leq \bar{R} \leq \bar{x} \\ 0 - e^2/2 & \text{if } \bar{x} < \bar{R} \end{cases}$$

Thus, the solution of the problem Max $C(e, \bar{R})$ is

$$e^*(\bar{R}) = \begin{cases} (\bar{x} - \underline{x}) & \text{if } \bar{R} < \underline{x} \\ (\bar{x} - \bar{R}) & \text{if } \underline{x} \leq \bar{R} \leq \bar{x} \\ 0 & \text{if } \bar{x} < \bar{R} \end{cases} \qquad (2)$$

because in its three steps $C(e, \bar{R})$ is a concave function of e (with a fixed $\bar{R}$)and the previous conditions are the first order conditions. Thus, (2) is the optimal strategy for the country.

When the country follows its optimal strategy, the expected utility for the bank is:

$$B(e^*(\bar{R}), \bar{R}) = E\, G(R) = \begin{cases} \bar{R} & \text{if } \bar{R} < \underline{x} \\ \underline{x} + e^*(\bar{R})(\bar{R} - \underline{x}) & \text{if } \underline{x} \leq \bar{R} \leq \bar{x} \\ \underline{x} \quad e^*(\bar{R})(\bar{x} - \underline{x}) & \text{if } \bar{x} < \bar{R} \end{cases}$$

Using (2) in the previous expression we obtain:

$$B(e^*(\bar{R}), \bar{R}) = \begin{cases} \bar{R} & \text{if } \bar{R} < \underline{x} \\ \underline{x} + (\bar{x} - \bar{R})(\bar{R} - \underline{x}) & \text{if } \underline{x} \leq \bar{R} \leq \bar{x} \\ \underline{x} & \text{if } \bar{x} < \bar{R} \end{cases} \qquad (3)$$

In the interval $\underline{x} \leq \bar{R} \leq \bar{x}$ the bank's objective function is concave and the first order condition is $(\bar{x} - \bar{R}) - (\bar{R} - \underline{x}) = 0$, from where $\bar{R} = M$ which yields $B(e(M), M) = \underline{x} + (\bar{x} - M)^2 > \underline{x}$, that is, greater than the values taken by the function in the regions $\bar{R} < \underline{x}$ and $\bar{R} > \bar{x}$.

The equilibrium effort is $e^*(M) = (\bar{x} - M)$ substituting M in (2). The first best effort is obtained when the country totally internalize the consequences of its decision, for instance when $\bar{R} = 0$, and $e^*(0) = (\bar{x} - \underline{x})$, substituting in (2). To see it formally, define $\underline{TC}$ ($\underline{TB}$) as the product received by the country (the bank) if $\underline{x}$ occurs, so that $\underline{x} = \underline{TC} + \underline{TB}$. Likewise, define $\overline{TC}$ ($\overline{TB}$) as the product received by the country (the bank) if $\bar{x}$ occurs, so that $\bar{x} = \overline{TC} + \overline{TB}$. Maximize now the expected utility for the country, $\underline{TC} + e\,(\overline{TC} - \underline{TC}) - e^2/\,2$, subject to guarantee a fixed expected utility for the bank, $\bar{B} = \underline{TB} + e\,(\overline{TB} - \underline{TB})$. We can now write this restriction as $\bar{B} = \underline{TB} + e\,(\bar{x} - \underline{x} - (\overline{TC} - \underline{TC}))$, from where, solving for $(\overline{TC} - \underline{TC})$ and substituting in the objective function we transform the problem in the maximization of $\underline{x} + e\,(\bar{x} - \underline{x}) - e^2/\,2 - \bar{B}$, which yields $e^{fb} = (\bar{x} - \underline{x})$.

■

2. A DYNAMIC MODEL

The model

In this section we model a repeated relationship. To simplify, we consider a two-period model. One question we address in this dynamic model is the intertemporal nature of the forgiveness scheme that finds optimal to set the bank. The assumptions we make about its capacity of commitment play a key role. We assume that the bank can fix, at the beginning of the game, an upper bound on the amount received in the first period, and that it can also make a commitment about the upper bound for the second period. This second commitment can be made contingent on the amount the country repays in the first period. This makes sense under the (realistic) hypothesis that the payments made by the country can be verified by outside observers. At the beginning of the second period the bank can, nevertheless, make a further reduction in the outstanding debt if it wishes to do so.

In each period t a random product x_t is obtained. x_t has as support the set $\{ \underline{x} , \bar{x} \}$ and depends on the country's effort in period t, e_t, and on a random state of nature. Formally, given e_1 and e_2 , x_1 and x_2 are independent random variables with (marginal) distribution determined by

$$P(x_t = \bar{x}) = e_t \qquad (1')$$

To describe the first action of the bank we define $\bar{R}^1_2(\cdot)$ as a function from $\mathbb{R}$ into $\mathbb{R}$ which assigns to each amount paid by the country in the first period, an upper bound with which the second period starts. At the beginning of the first period the bank sets an upper bound $\bar{R}^1_1 \in \mathbb{R}$ to the amount paid by the country in this period and a function $\bar{R}^1_2(\cdot)$. This second function allows the bank to commit to an upper bound $\bar{R}^1_2$ for the second period, whose value depends on the amount it receives in the first period. Both bounds are legally binding. Next, in I.2, knowing the commitments made by the bank in I.1, the country makes an effort $e_1 \in [0, 1]$. In I.3 nature chooses a product according to (1') and the bank receives $R_1 = \min\{ \bar{R}^1_1, x_1 \}$. This is the end of the first period. In II.1, knowing the amount R_1, the bank chooses an upper bound $\bar{R}^2_2 \in \mathbb{R}$ that satisfies $\bar{R}^2_2 \leq \bar{R}^1_2(R_1)$, that is, we allow it to make an additional reduction in the second period debt. Knowing the new bound $\bar{R}^2_2$ the country chooses $e_2 \in [0, 1]$ in II.2. Finally, in II.2 nature acts producing x_2 according to (1') and the bank receives $R_2 = \min\{ \bar{R}^2_2, x_2 \}$
The timing of the model is shown in the following figure.

Figure 2.

I.1	I.2	I.3	II.1	II.2	II.3
bank sets $\bar{R}^1_1$, $\bar{R}^1_2(\ )$	country chooses e_1	realization of x_1	bank sets $\bar{R}^2_2$	country chooses e_2	realization of x_2

Agents' utilities are as follows

country $\quad H(x_1 - R_1, e) + b\ H\ (x_2 - R_2, e_2)$

bank $\quad G(R_1) + b\ G(R_2)$

where b is a discount factor satisfying $0 < b < 1$.

Equilibrium concept

To define the strategies at the disposal of each agent we first define the histories observed at each date (remember that we assume that everything is observed by both agents. It is the nonverifiability of effort which creates a problem)

Let $h_{I.1} = 0$, $h_{I.2} = (\bar{R}^1_1, \bar{R}^1_2(R_1))$, $h_{I.3} = (h_{I.2}, e_1)$

$h_{II.1} = (h_{I.3}, x_1, R_1)$, $h_{II.2} = (h_{II.1}, \bar{R}^2_2)$, $h_{II.3} = (h_{II.2}, e_2)$

be the histories observed up to date t.r, t = I,II,r = 1, 2, 3.

A strategy for the bank specifies, for the first period, an upper bound $\bar{R}^1_1$ and a function $\bar{R}^1_2(\cdot)$ and, for the second period, a function $\bar{R}^2_2(\cdot)$ that assigns to each history $h_{II.1}$ an upper bound $\bar{R}^2_2$ satisfying the condition $\bar{R}^2_2(h_{II.1}) \leq \bar{R}^1_2(R_1)\ \forall\ R_1$. We denote a bank's strategy as

$$R = ((\bar{R}^1_1, \bar{R}^1_2(R_1)),\ \bar{R}^2_2(h_{II.1}))$$

It is important to note that although $\bar{R}^2_2$ can depend, in the bank strategy, of all the history observed up to II.1, it is legally binding only since it is officially announced in II.1[4]

[4]Note the different kind of dependence of $\bar{R}^1_2$ and $\bar{R}^2_2$ with respect to

A strategy for the country specifies, for the first period, a function $e_1^2(\cdot)$ which assigns to each history $h_{I.2}$ an effort level at I.2 and, for the second period, a function $e_2^2(\cdot)$ which assigns to each history $h_{II.2}$ an effort level at II.2. We denote the country's strategy as $\mathbf{e} = (e_1^2(\cdot), e_2^2(\cdot))$

We are interested in characterizing the self enforceable contracts, that is, those which agents respect because it is in their interest to do so (the only commitment we allow consists in the upper bounds $\bar{R}_t^s$ being legally binding from the date they are set, s).

So, we will ask each agent's strategy not only to maximize its expected utility given the other agent's strategy, but also to satisfy this condition for each history of the game, that is, the continuation of a strategy should prescribe an optimal behavior given the continuation of the other agent's strategy. So, we will require subgame perfection.

To formalize this idea we will define the expected utility at each date as a function of the observed histories and the continuation of the strategies.

Let $B_t(\mathbf{R}, \mathbf{e}, h_{t.1}) = E_{t.1} \sum_{j=t}^{2} b^{t-j} G(R_j)$

be the expected utility of the bank, evaluated in t.1, over the two period horizon, when the pair of strategies $\mathbf{R}$, $\mathbf{e}$ are followed from t.1, and the history $h_{t.1}$ has been observed.

Let $C_t(\mathbf{R}, \mathbf{e}, h_{t.2}) = E_{t.2} \sum_{j=t}^{2} b^{t-j} H(x_j - R_j)$

R_1:

Announcing the function $\bar{R}_2^1(R_1)$ is part of the first period bank's action (this action also includes the announcement of $\bar{R}_1^1$). The bank can commit not to charge tomorrow more than $\bar{R}_2^1$ if it receives R_1 today. In contrast, $\bar{R}_2^2$, the second action of the bank, is only a number. Now, every action, in the bank's strategy, can depend on everything the bank has observed when it makes it. In this sense, in the bank's strategy, $\bar{R}_2^2$ depends on R_1.

be the expected utility for the country defined in the same way, except that we use t.2 instead of t.1.

An optimal self enforcing contract for the two-period model (OSEC2) is a pair of strategies $\mathbf{R}^*$, $\mathbf{e}^*$, such that

(i) $\mathbf{R}^* \in \underset{\mathbf{R}}{\operatorname{argmax}} \quad B_1(\mathbf{R}, \mathbf{e}^*, h_{I.1})$

(ii) $\mathbf{R}^* \in \underset{\mathbf{R}}{\operatorname{argmax}} \quad B_2(\mathbf{R}, \mathbf{e}^*, h_{II.1}) \quad \forall\ h_{II.1}$

(iii) $\mathbf{e}^* \in \underset{\mathbf{e}}{\operatorname{argmax}} \quad C_1(\mathbf{R}^*, \mathbf{e}, h_{I.2}) \quad \forall\ h_{I.2}$

(iv) $\mathbf{e}^* \in \underset{\mathbf{e}}{\operatorname{argmax}} \quad C_2(\mathbf{R}^*, \mathbf{e}, h_{II.2}) \quad \forall\ h_{II.2}$

This definition tells us that each time an agent acts, for each observed history, its strategy must prescribe an optimal behavior given the behavior the other agent's strategy prescribes from that moment on:
(i) requires the bank's strategy to be its best response to the country's strategy and when restricting it to (ii) we guarantee an optimal behavior in II.1. Likewise, (iii) asks the country to use its best response in I.2 and (iv) makes this response 'credible' in II.2

3. THE BANK'S STRATEGY

After verifying the existence of an OSEC2 and the uniqueness in expected utilities we characterize the bank's strategy.

In a one-period model if the bank wants to forgive debt it has a unique way to do it. Its only decision variable is $\bar{R}$. We have seen that it chooses $\bar{R} < \bar{x}$, in spite of being able to choose $\bar{R} = \bar{x}$, to give incentives for the country to make a positive effort. In a two-period model there are more ways to forgive. We find that more than one way is optimal. To characterize the bank's strategies that can be in an OSEC we proceed in two steps. First we asses that it is optimal to place all the incentives in the future. This will completely determine an OSEC2. Then we see that setting $\bar{R}^1_1 < \bar{x}$, that is, promising to charge less product than the available today's product, is optimal only if in doing so no overhang is left for tomorrow (iii.ii). In particular, duplicating the one-period model, $\bar{R}^1_1 = M$, $\bar{R}^1_2(R_1) = M\ \forall\ R_1$, is not optimal. To see why, notice that a small change of this schedule that rises $\bar{R}^1_1$ and reduces $\bar{R}^1_2(R_1)$ only in $R_1 = \bar{R}^1_1$ would modify today a zero sum's sharing problem while the smaller $\bar{R}_2(\bar{R}_1)$ would put us closer to the first best effort level in the last period. Intuitively, if the problem is that the debt overhang creates disincentives, the optimal way to forgive involves not leaving overhang for tomorrow if it can be exchanged for today's product. As long as $\bar{R}^1_2(\bar{R}^1_1) \leq \underline{x}$, $\bar{R}_1$ and $\bar{R}_2(\bar{R}_1)$ can be exchanged on a b units of $\bar{R}_1$ for one unit of $\bar{R}_2(\bar{R}_1)$ basis in the optimal contract. Its sum (in present value) is the total repayment in case $x_1 = x_2 = \bar{x}$.

Since $\bar{R}_2(\bar{R}_1) < \bar{x}$, it continues to be true that, as in the one-period model, the bank partially relinquishes its rights on the country's product. (Moreover, $\bar{R}_1 + b\ \bar{R}_2 < M(1+b)$, that is, the discounted value of the highest repayments the bank can get is smaller than the discounted value of the highest repayment it would get by duplicating the one period solution). On the other hand, the bank punishes a small x_1 as severely as the credibility of the punishment allows it to - ($\bar{R}^2_2(h_{II.1}) = M$ if $x_1 = \underline{x}$ - and would prefer punishing even more severely such outcome, that is, setting $\bar{R}^1_2(\underline{x}) > M$ and committing not to reduce that bound later on.

The bank's strategy discards, due to credibility arguments, non-binding promises, such as $\bar{R}^1_2(\bar{x}) = M$, but $\bar{R}^2_2(h_{II.1}) < \underline{x}$ if $x_1 = \bar{x}$

The following proposition guarantees the existence of an OSEC2 and the uniqueness in expected utilities and characterizes the bank's strategy.

Proposition 2

(i) The expected utilities for the bank and the country and the observed efforts are the same under every OSEC2

(ii) There exists an OSEC2 in which the bank's strategy satisfies:

$$\bar{R}_1^{1*} = \bar{x}$$

$$\bar{R}_2^{1*}(\underline{x}) = M, \; \bar{R}_2^{1*}(\bar{x}) < \underline{x}$$

$$\bar{R}_2^{2*}(h_{II.1}) = \bar{R}_2^{1*}(R_1) \; \forall \; R_1$$

(iii) The bank's strategy is unique except for the following two changes:

(iii.i) $\bar{R}_2^1(\underline{x}) > M$ *and* $\bar{R}_2^2(h_{II.1}) = M$ *if* $x_1 = \underline{x}$.

(iii.ii) $\bar{R}_1^1 + b\,\bar{R}_2^1(\bar{R}_1^1) = \bar{R}_1^{1*} + b\,\bar{R}_2^{1*}(\bar{R}_1^{1*})$ *and* $\bar{R}_2^1(\bar{R}_1^1) \leq \underline{x}$

(iv) If we take from the bank the chance of reducing $\bar{R}_2^1$ *in II.1, it obtains an expected utility strictly greater.*

Proof. (See appendix)

The bank's strategy as a one-period maturity debt

We have written the bank's strategy as upper bounds for each period. In equilibrium it is possible to see it as a debt with maturity one period. When we do it, we find that it consists of an initial debt reduction (we start from a debt that allows the bank to withdraw the entire product always) and a future reduction contingent on the first period outcome. To be precise, the future reduction is made if and only if the first period product is 'small'. Intuitively, at each point in time there is an optimal level of one period debt. If a low product occurs, the outstanding debt will be so high that it will be in the interest of the bank to reduce it again.

Proposition 3

There exists an optimal strategy for the bank which can be written as a debt D_1^{2*} which is due in the first period, and whose unpaid part grows at a rate 1/ b. Since $D_1^{2*} > \bar{x}$, the bank can confiscate x_1 in its entirety and fix $(D_1^{*} - x_1)/b$ as the second period obligation. D_1^{2*} satisfies:

i) $[D_1^{2*} - \underline{x}] / b > M$

ii) $[D_1^{2*} - \bar{x}] / b < \underline{x}$

iii) $D_1^{2*} < M (1 + b)$

Proof

Consider the bank's strategy that sets $\bar{R}_1^{*} = \bar{x}$, and define $D_1^{*} = b\ (\underline{x} + 5(\bar{x} - M)^2/4) + M$. Then we have $\underline{x} + b\ M < D_1^{*} < \bar{x} + b\ \underline{x}$

■

That is, at the beginning of the game a debt is set that involves a partial forgiveness. It is a one period maturity debt an it is greater than $\bar{x}$, so that it allows the bank to withdraw x_1 in its entirety. The unpaid part of the debt grows at a rate 1/ b and the level of D_1^{2*} is such that if $x_1 = \bar{x}$, at the beginning of the second period we will have a debt smaller than $\underline{x}$ and the bank will not forgive any more; but if $x_1 = \underline{x}$, the outstanding debt will be so high that the bank will make an additional forgiveness. Let us remark that if $x_1 = \underline{x}$, a further reduction is called for, whereas if $x_1 = \bar{x}$, the outstanding debt, $(D_1^{2*} - \bar{x})/ b < \underline{x}$, is repaid without further reduction. Repeating a partial forgiveness can be a rational action on the part of the bank. We can interpret its actions as an adjustment made each period of the debt the country owes to it, so as to keep this debt at an optimal level. If $x_1 = \underline{x}$ and the bank did not forgive once more, the resulting amount of debt would be too big.

4. THE EQUILIBRIUM LEVEL OF EFFORT

We find that the equilibrium first period effort is bigger than that of the one-period model. The intuition behind it is that in solving the trade off between receiving more product if $\bar{x}$ happens and giving the proper incentives to the country, the bank distributes between both of them the benefits of obtaining $\bar{x}$ instead of $\underline{x}$. In a one-period model, these benefits consist of the difference $\bar{x} - \underline{x}$. In a two-period model, we must add the gains from eliminating the debt overhang and its associated efficiency loss in the last period. Due to the preceding feature, as we will see later on, there may be situations in which the debt overhang leads to an equilibrium effort greater than the first best equilibrium.

Since we have already described the bank's strategy, it is clear what the second period equilibrium efforts look like. We will have the first best level if the first period product is high and the level arising from an infinite debt otherwise.

Proposition 4

Every optimal self-enforceable contract in the two-period model induces the following equilibrium efforts:

$$e_1^{2*}(R^*) = (\bar{x} - M) + b\,(\bar{x} - M)^2/\,4 \;>\; e^*(R^*) = (\bar{x} - M)$$

$$e_2^{2*}(h_{II.1}) \;=\; e^{fb} = (\,\bar{x} - \underline{x}\,) \text{ if } x_1 = \bar{x}$$

$$e_2^{2*}(h_{II.1}) \;=\; e_1^{1*} = \;(\,\bar{x} - M\,) \text{ if } x_1 = \underline{x}$$

Proof

The equilibrium efforts can be computed using the strategy defined by (2) in the proof of proposition 1 with $\bar{R} = \bar{R}_2^2$ (for $e_2^{2*}(\cdot\ )$) and (6) in the proof of proposition 2 (for $e_1^{2*}(\ )$).

■

5. DEBT REDUCTION VS. TEMPORARY RELIEF

We have seen (proposition 3) that the optimal policy for the bank can be written as an initial debt reduction and a future reduction contingent in the first period outcome. We will compare this strategy with the optimal strategy in a restricted opportunity set.

To motivate the restriction we impose, let us remember that during the earliest years of the debt crisis, the policy to face the problem was one of temporary relief, of reschedulings consisting of reducing the repayments the country should meet in the short term, moving their maturity towards the future.

Bearing in mind this fact as a motivation, assume that in the game we haveanalyzed we restrict the bank to a temporary relief policy. Beginning with an infinite debt, the bank decides only how much relief to give to the country in the current period, so that we start the second period with an infinite debt once more. Formally, let us impose[5] $\bar{R}_2^1(R_1) \geq \bar{x}$ $\forall$ R_1 as a restriction in the bank's first action. By a temporary relief policy we mean a policy that satisfies this restriction. It is easily seen that in this context the optimal strategy for the bank is to set twice the optimal one period optimal bound because there is no link at all between the different periods. In each period the bank induces the country to make some effort by promising not to withdraw more than M in the current period. We know this policy is not optimal from the bank's perspective when it can choose $\bar{R}_2^1(R_1)$. The following proposition tells us that this policy also reduces the welfare of the country, relative to an optimal policy without restrictions.

Proposition 5

Under the optimal self-enforcing contract in the two-period model both the bank and the country obtain an expected utility strictly bigger than that resulting from restricting the bank to choose

$\bar{R}_2^1(R_1) \geq \bar{x} \quad \forall R_1$

Proof

It only remains to be seen that the expected utility for the country under the two period optimal self-enforcing contract is bigger than the expected utility resulting from getting twice the one period optimal contract. We have

EU country under repetition = $b\,(\bar{x} - M)^2 / 2 + (\bar{x} - M)^2 / 2$

EU country under osec2 = $b\,(\bar{x} - M)^2 / 2 + (\bar{c}^* - \underline{c}^*)^2$

EU country under osec2 - EU country under repetition = $(e_1^{2*})^2 - (e_1^{1*})^2$.

Thus, the expected utility for the country is (strictly) greater under osec2 if and only if its first period effort under osec2 is (strictly) grater.

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[5] The reader will notice that imposing $\bar{R}^1(R) \geq M$ would lead us to the same results

6. PARTIAL FORGIVENESS IN DYNAMIC MODELS

One of the questions we wanted to address was that of the robustness of the argument favoring voluntary debt reduction to adding periods to the length of the relationship. We find that the argument is reinforced.

Definition. We will say that there is debt forgiveness if $\bar{R}^s_t < \bar{x}$ for some(s,t), $s \leq t$

Proposition 6

(i) If there is debt forgiveness in the one period model, then there is debt forgiveness in the two-period model.
(ii) There exist functions $P(\bar{x}|e)$ increasing and affine in e and g(e) increasing and convex, such that in the one period model there is no debt forgiveness, but in the two-period model there is.

Proof

(i) it is straightforward: $\bar{R}^2_2(R_1) \leq \bar{R}^* \;\forall\; R_1$ because at the beginning of the second period the bank will reduce the upper bound it finds if it is greater than the optimal one period upper bound.

(ii) Consider

$$P(\bar{x}|e) = \begin{cases} 0 & \text{si } e \leq 1/2 \\ e + 1/2 & \text{si } e \in [0, 1/2] \\ 1 & \text{si } 1/2 < e \end{cases}$$

$$g(e) = e^2 / 2$$

Then, following exactly the same steps as in the case we have studied, we find that in the one-period model a necessary condition for equilibrium is $\bar{R}^1_1 = \bar{x}$. Nevertheless, if in the two periods model we set $\bar{R}^1_1 = \bar{R}^1_2(\underline{x}) = \bar{x}$, then a necessary condition for equilibrium is $\bar{R}^1_2(\bar{x}) < \bar{x}$

■

To see why it can be that in the one-period model the bank does not forgive debt and yet it does it in the two-period model, think of the following extreme situation. The functional forms are such that from a static viewpoint the bank is indifferent between forgiving debt and not doing it, that is, it finds optimal to leave a level just equal to $\bar{x}$ in the one-period model. Then, in the two-period model, from the perspective of the first period, a small debt reduction for the second period has only a second order effect. Since it has a first order effect on the country's initial effort, it clearly is welfare improving.

7. LIMITED INITIAL DEBT

Up to now we have taken as a starting point a situation in which the bank's legal rights allow it to withdraw the entire country's product in each period. In this section we analyze a situation in which the bank's legal rights are narrower.

We assume the same production technology and utility functions as in the previous section except that, for simplicity, we take discount factors equal to one for both agents.

The country's initial legal obligations towards the bank are described as follows. $\bar{R}_1^{IN} \in \mathbb{R}$ is the amount the country must pay in the first period. There is also a function $\bar{R}_2^{IN}(\cdot)$ from $\mathbb{R}$ into $\mathbb{R}$ which assigns to each amount paid in the first period the debt with which the second period starts. This specification is very general. It allows as a special case, for instance, one in which the entire debt is due in the first period and any unpaid amount becomes an obligation for the second period at a given interest rate. We assume that the bank can make to the country a "take it or leave it" offer. It can propose a pair $\bar{R}_1^1$, $\bar{R}_2^1(\cdot)$ which will substitute the pair $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(\cdot)$ if and only if the country accepts it. Except for this change, the model is equal to the one in the previous section. The description of the game between the bank and the country is as follows.

At the start of the game, in I.1, the bank proposes an upper bound for the first period $\bar{R}_1^1 \in \mathbb{R}$ and a function $\bar{R}_2^1(\cdot)$ from $\mathbb{R}$ into $\mathbb{R}$ which assigns an upper bound for the second period as a function of the payment made in the first period. Next, in I.1 bis, the country accepts ($A_1 = 1$) or rejects ($A_1 = 0$) the bank's proposal. In the former case the bounds proposed in I.1 substitute the initial ones. In the second case they do not. In I.2, knowing the bounds resulting from the above process, the country makes an effort $e_1 \in [0, 1]$. This effort parameterizes the distribution of x_1 which is realized and divided in I.3. This marks the end of the first period.

In II.1 the bank can give an additional forgiveness and set $\bar{R}_2^2$ smaller than the bound established in I.1 bis. To do this the bank does not need the approval of the country: it can renounce to a previous right. This decision is legally binding. In II.2, knowing $\bar{R}_2^2$, the country chooses $e_2 \in [0, 1]$ which parameterizes the distribution of x_2, realized and shared in II.3.
The time line for choices is as shown in figure 3

Figure 3

I.1	I.1bis	I.2	I.3	II.1	II.2	II.3
bank proposes $\{\bar{R}^1_1, \bar{R}^1_2(\ )\}$	country can reject proposal	country makes e_1	realization of x_1	bank chooses $\bar{R}^2_2$	country makes e_2	realization of x_2

<u>Equilibrium concept</u>

Let

$\mathbf{h}_{I.1bis} = (\bar{R}^1_1,\ \bar{R}^1_2(\cdot)\ ),\ \mathbf{h}_{I.2} = (\mathbf{h}_{I.1bis},\ A_1),\ \mathbf{h}_{I.3} = (\mathbf{h}_{I.2},\ e_1),$

$\mathbf{h}_{II.1} = (\mathbf{h}_{I.3},\ x_1,\ R_1),\ \mathbf{h}_{II.2} = (\mathbf{h}_{II.1},\ \bar{R}^2_2)$

be the observed histories at date t.r, $t \in \{\ I,\ II\ \}$, $r \in \{\ 1,\ 1bis,\ 2,\ 3\ \}$

A strategy for the country specifies a function $A_1(\cdot)$ which assigns to each history $h_{I.1bis}$(consisting of a bank's proposal) an acceptance ($A_1(\cdot) = 1$) or rejection ($A_1(\cdot) = 0$) decision, a function $e_1(\cdot)$ which assigns to each history $\mathbf{h}_{I.2}$ an effort level for the first period, and a function $e_2(\cdot)$ which assigns to each history $\mathbf{h}_{II.2}$ an effort level for the second period. We denote a country's strategy as

$$\mathbf{e} = (\ e_1(\cdot), e_2(\cdot), A_1(\cdot))$$

A strategy for the bank specifies, for the first period, a bound $\bar{R}^1_1$ and a function $\bar{R}^1_2(\cdot)$ and, for the second period, a function which assigns to each history $h_{II.1}$ a bound $\bar{R}^2_2$ that satisfies $\bar{R}^2_2(h_{II.1}) \leq \bar{R}^1_2(R_1)\ \forall\ R_1$. We denote a bank's strategy as

$$\mathbf{R} = (\bar{R}^1_1,\ \bar{R}^1_2(\cdot), \bar{R}^2_2(\cdot))$$

Let $B_{t.r}(\mathbf{R},\ \mathbf{e},\ \mathbf{h}_{t.r})$ and $C_{t.r}(\mathbf{R},\ \mathbf{e},\ \mathbf{h}_{t.r})$ be the expected utilities for the bank and for the country, evaluated at date t.r, when the history $\mathbf{h}_{t.r}$ has been observed and from that moment on the strategies $\mathbf{R}$,$\mathbf{e}$. are followed.

An OSEC with initial bounds $\bar{R}^{IN}_1$, $\bar{R}^{IN}_2(R_1)$ is a pair of strategies, $\mathbf{R}^*$,$\mathbf{e}^*$, such that:

a) $\mathbf{lR}^{*} \in \underset{\mathbf{lR}}{\operatorname{argmax}} \; B_{t.r}(\mathbf{lR}, \mathbf{le}, h_{t.r})$ $\forall$ t.r in which the bank acts, $\forall\, h_{t.r}$

b) $\mathbf{le}^{*} \in \underset{\mathbf{le}}{\operatorname{argmax}} \; C_{t.r}(\mathbf{lR}, \mathbf{le}, h_{t.r})$ $\forall$ t.r in which the country acts, $\forall\, h_{t.r}$

It will be helpful to define a situation in which the initial bounds remain in place (even if it is not optimal), and in the subsequent dates agents act optimally.

A continuation OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$ is a pair of strategies $\mathbf{cR}^{*}$, $\mathbf{ce}^{*}$, such that in I.1 the bank proposes the initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$, and:

a) $\mathbf{cR}^{*} \in \underset{\mathbf{cR}}{\operatorname{argmax}} \; B_{II.1}(\mathbf{cR}, \mathbf{ce}, h_{II.1})$

b) $\mathbf{ce}^{*} \in \underset{\mathbf{ce}}{\operatorname{argmax}} \; C_{t.r}(\mathbf{cR}, \mathbf{ce}, h_{t.r})$ $\forall$ t.r in which the country acts, $\forall\, h_{t.r}$

Analysis. The welfare of the country and an optimal first period bound

What sort of proposal will the bank make and how will the country react? We start to answer to this question in the following proposition, which establishes two features of the optimal contract.

As a consequence of giving the bank the right to make "take it or leave it" offers, we will have the country willing to accept every proposal that leaves it with the same expected utility as the initial bounds (notice that if the initial bounds remain in place and then agents act optimally uniqueness in the country's expected utility is obtained). This is exactly the expected utility the country will obtain with one exception: when the initial bounds are such that, if left, would lead to an expected utility for the bank smallerthan the one it obtains in the equilibrium of the unlimited debt case. On the other hand, it remains true that the bank finds it convenient to withdraw all today's product.

Proposition 7

Let c_1^{min} be the expected utility obtained by the country under OSEC2 (as in propositions 2 and 3, when the bank's initial rights are unlimited).

Let c_1^{cont} be the expected utility obtained by the country under a continuation OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$.

Let $c_1 = \max\{ c_1^{min}, c_1^{cont}\}$

Then:

i) Under every OSEC with initial bounds $\bar{R}_1^{IN}, \bar{R}_2^{IN}(R_1)$, the expected utility for the country is equal to c_1

ii) There exists an OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$ in which $\bar{R}_1^1 = \bar{x}$

Furthermore, if there exist an OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$ in which $\bar{R}_1^1 < \bar{x}$, then it also exist an OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$ in which $\bar{R}_1^1 = \bar{x}$ with the same first period effort for the country.

Proof (See appendix)

The bounds for the second period and the equilibrium effort

In this section we analyze the two remaining variables in the bank's strategy and the resulting country's effort.

The bank must choose $\bar{R}_2^1(\underline{x})$ and $\bar{R}_2^1(\bar{x})$ such that they provide the country with an expected utility equal to c_1. We are interested in finding the conditions under which the bank punishes as hard as credible an $x_1 = \underline{x}$ realization, and in the induced behavior of e_1.

It turns out that the answers depend on the size of ($\bar{x} - \underline{x}$). There are,nevertheless, some invariant features If the initial bounds lead us to a situation close to that of unlimited debt, in terms of the country's expected utility, then $\bar{R}(\ \underline{x}\ )= M$, that is, the bank punishes as much as it is credible a bad x_1 realization.

Consider now the other polar case. Let c_1^{max} be the expected utility for the country under the initial bounds $\bar{R}_1^{IN} = \underline{x}$, $\bar{R}_2^{IN}(R_1) \equiv \underline{x}$. For c_1 values close to c_1^{max}, $\bar{R}_2^1(\ \underline{x}\ )$ is close to $\underline{x}$.

The behavior of e_1 also exhibits this continuity in the extreme values of c_1: if c_1 is close to c_1^{min} e_1 will be close to $e_1(c_1^{min})$ and if c_1 approaches c_1^{max}, we will approach to the first best effort.

A thorough characterization of the OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$

for $0 < (\bar{x} - \underline{x}) < 1$ is shown in the appendix. In this section we state the results for a "very small" $(\bar{x} - \underline{x})$ and mention in a rough way the results for other values of $(\bar{x} - \underline{x})$.

In the very small case, if the initial bounds lead us to a situation close to the one with unlimited debt, in terms of expected utility for the country, that is, in a neighborhood of c_1^{min}, $\bar{R}_2^1(\underline{x})$ is as big as the credibility allows it to but, if we consider increasingly bigger values of c_1, we arrive to a point in which the bank starts to reduce $\bar{R}_2^1(\underline{x})$, and it does it continuously. Then the equilibrium $\bar{R}_2^1(\underline{x})$ is first constant and then strictly decreasing in c_1. As for the effort, we find that as we go deeper into a debt overhang situation, the first period effort gets smaller, that is, if starting from a no-debt-overhang situation, in which the country repays $\underline{x}$ each period and makes the first best effort, we consider ever smaller values for the country's expected utility (ever bigger bank's rights), then the equilibrium effort becomes increasingly smaller (and it does it continuously) until we reach the unlimited debt equilibrium effort.

For other $(\bar{x} - \underline{x})$ values, we have roughly the following results: If $(\bar{x} - \underline{x})$ is 'median' we cannot guarantee $e_1(c_1)$ to be monotone in c_1, but at least we keep the property of having $e_1 < (\bar{x} - \underline{x})$ always.

This feature will disappear if we consider even bigger values for $(\bar{x} - \underline{x})$. It will disappear for instance when c_1 is such that the bank still finds convenient to set $\bar{R}_2^1(\underline{x}) = M$ (to punish a first period $\underline{x}$ realization the way it does under unlimited debt) and to provide the augmented expected utility $c_1 > c_1^{min}$ through, solely, a reduction in $\bar{R}_2^1(\underline{x})$.

If we consider even bigger values of $(\bar{x} - \underline{x})$, we find a new feature. We loose the continuity of $\bar{R}_2^1(\underline{x})$ and e_1 in c_1. The objective function is not cuasiconcave and this causes $\bar{R}_2^1(\underline{x})$ and e_1 to take on two different values for a critical c_1 value. Nevertheless, in all but this point those variables exhibit a continuous behavior in c_1.

Finally, for $(\bar{x} - \underline{x})$ values close to one, since the specification of the model restricts us to $e \leq 1$, there are c_1 values (to be precise there is an interval) in which the country's first period effort equals unity, and both $\bar{R}_2^1(\underline{x})$ and e_1 move such that this restriction is satisfied with equality.

Proposition 8

There exists $(\bar{x} - \underline{x})^{pp}$ *such that if* $(\bar{x} - \underline{x}) \leq (\bar{x} - \underline{x})^{pp}$, *then the OSEC with initial bounds* $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$ *in which* $\bar{R}_1^1 = \bar{x}$ *satisfies:*

i) (Continuity of $\bar{R}^1(\underline{x})$ *and* e_1*)*

There exists c_1^*, $c_1^{**} \in (c_1^{min}, c_1^{max})$ *s.t.* $\bar{R}_2^1(\underline{x})$ *and* e_1 *are continuous and (except in* c_1^**) differentiable functions of* c_1*, with domain* $[c_1^{min}, \infty)$.

ii) (Behavior of $\bar{R}_2^1(\underline{x})$*)*

$\bar{R}^1(\underline{x})$ *satisfies:*

$c_1^{min} \leq c_1 \leq c_1^* \Rightarrow \bar{R}_2^1(\underline{x}) = M$

$c_1^* < c_1 < c_1^{max} \Rightarrow \underline{x} < \bar{R}_2^1(\underline{x}) < M$ *and* $\bar{R}_2^1(\underline{x})$ *is strictly decreasing in c*

$c_1^{max} \leq c_1 \Rightarrow \bar{R}_2^1(\underline{x}) \leq \underline{x}$

iii) (Behavior of $e_1(c_1)$*)*

$e_1(c_1)$ *is strictly increasing in the interval* $c_1 \in [c_1^{min}, c_1^{max}]$*, where* $e_1(c_1^{min})$ *is the equilibrium effort in the unlimited debt model, and* $e_1(c_1) = e^{fb} = (\bar{x} - \underline{x})$ *if* $c_1^{max} \leq c_1$

8. DIFFERENT DISCOUNT FACTORS

We now consider the case in which the bank and the country have different discount factors.

All the definitions in section 1.2 apply, except for the following utility functions:

country's utility: $x_1 - R_1 - (e_1)^2/2 + b^C [x_2 - R_2 - (e_2)^2/2]$

bank's utility: $R_1 + b^B R_2$

If $b^B < b^C$, it remains optimal to place all the incentives to effort tomorrow. This is intuitively reasonable, because the country is more patient than the bank. In this case the inequality in the discount factors reinforces the reasons to place the "prize" tomorrow. As for the induced effort, it is bigger than the one exhibited by the one period model. This feature is reinforced. We find that if the inequality in the discount factor is big enough, the first period effort is greater than the first best level in the one period model. The reason is that the prize will be paid tomorrow and the future is very differently valued by both agents. The bank gives a prize to which it attaches a small value , but which is very valuable for the country. Nevertheless, the relevant comparison is with the first best effort in a two period model with $b^B < b^C$. When making this comparison the undereffort remains.

If $b^B > b^C$, it is not optimal to place all the incentives in the future. The country is less patient than the bank. It attaches more value (than the bank) to the current output. This renders optimal to give some relief today even though some overhang is left for tomorrow. The induced efforts do not have qualitative changes as compared with those of the equal discount factors model.

The following proposition formalizes the previous comments.

Proposition 9

i) If $b^B < b^C$, then

i.i) (strategy for the bank) there exists an OSEC2 in which

$$\bar{R}^1_1 = \bar{x}\ ,\ \bar{R}^1_2(\underline{x}) = M,\ \bar{R}^1_2(\bar{x}) < \underline{x}\ ,\ \bar{R}^2_2(h_{II.1}) = \bar{R}^1_2(R_1)\ \ \forall\ R_1$$

i.ii) (strategy for the country) under every OSEC2

$$e^{2*}_1 > e^{1*}_1$$

$$x_1 = \bar{x} \Rightarrow e^{2*}_2 = e^{2fb}_2 = (\bar{x} - \underline{x})$$

$$x_1 = \underline{x} \Rightarrow e^{2*}_2 = e^{1*}_1 = (\bar{x} - M)$$

Furthermore, there exists Θ ($\Theta > 1$), such that if $b^C / b^B > \Theta$, then $e^{2*}_1 > (\bar{x} - \underline{x})$ but if $e^{2*}_1 \in (0,1)$, then $e^{2*}_1 < e^{2fb}_1 = \max\{(\bar{x} - x)\, b^c / b^B, 1\}$

ii) If $b^C < b^B$, then

ii.i) (strategy for the bank) there exists an OSEC2 in which

$$\bar{R}^1_1 < \bar{x}\ ,\ \bar{R}^1_2(\underline{x}) = M,\ \bar{R}^1_2(\bar{R}^1_1) \in (\underline{x}\ ,\ \bar{x}),\ \bar{R}^2_2(h_{II.1}) = \bar{R}^1_2(R_1)\ \forall\ R_1$$

ii.ii) (strategy for the country) under every OSEC2

$$e^{2*}_1 > e^{1*}_1$$

$$e^{1*}_1 = e^{2*}_2(\underline{x}) < e^{2*}_2(\bar{x}) < e^{2fb}_2 = (\bar{x} - x)\, b^B / b^c$$

Proof (See appendix)

Remark. Notice that (as is also proven in the appendix) when the discount factors are different, the first period first best efforts are $e^{2fb}_1 = \max\{(\bar{x} - \underline{x})\, b^C / b^B, 1\}$, $e^{2fb}_2 = (\bar{x} - \underline{x})$ if $b^B < b^C$, and $e^{2fb}_1 = (\bar{x} - \underline{x})$, $e^{2fb}_2 = (\bar{x} - \underline{x})\, b^B / b^C$ if $b^C < b^B$.

Comparative statics

Consider a fix discount factor for the country and define the bank's discount factor as the reciprocal of the (gross) interest rate. In this section we address the question of what happens when the interest rate varies.

We obtain the following results. When the interest rate rises, the expected utility for the bank is reduced. The explanation is simple. It discounts more heavily its future earnings. This effect is of the first order. The adjustments made in the bank's decision variables -the bounds- are only of the second order. The expected utility for the country is risen. The optimal bounds set by the bank are parameters of the country's problem and so the adjustments these bounds suffer have a first order effect in the country's welfare. These adjustments improve the country's situation. When we compare two interest rates which are both in the region $1/(1+r) < b^C$ the idea is as follows. We have seen that in this case the bank places all the incentives to effort in the future. A rise in r makes then cheaper the cost of giving incentives to the county's effort and thus moves the bank to forgive more. This is in sharp contrast with the usual idea that a rise in r hurts the indebted countries. In a context of debt contracts with floating interest rates, an increase in this variable enhances the bank's rights over the country's product. But in our model this does not help the bank: anyway it already has more legal rights than it needs. A rise in r only means a rise in its impatience to be paid today. In equilibrium it already receives all the first period country's product. So its increased impatience moves it to urge the country to enhance the probability of a high product today. The optimal way to do it is promising an even lower repayment in the future.

Proposition 10

Let $b^B = 1/(1+r)$.
If $r_1 > r_2$ *then:*

i) The expected utility for the country under r_1 *is strictly greater than under* r_2
ii) The expected utility for the bank under r_1 *is strictly lower than under* r_2
iii) $e_1^{2*}(r_1) > e_1^{2*}(r_2)$

Proof. (See appendix)

9. A T-PERIOD MODEL

The model

Consider now a relationship that is repeated over T periods. The specification of the model is the natural extension of the two-period model. The production technology is as follows. x_1, x_2,...x_T are identical and independent random variables with

$$P(x_t = \bar{x}) = e_t^T \quad \text{for } t = 1, 2,\ldots T. \quad (1.T)$$

Next we describe the game between the bank and the country.

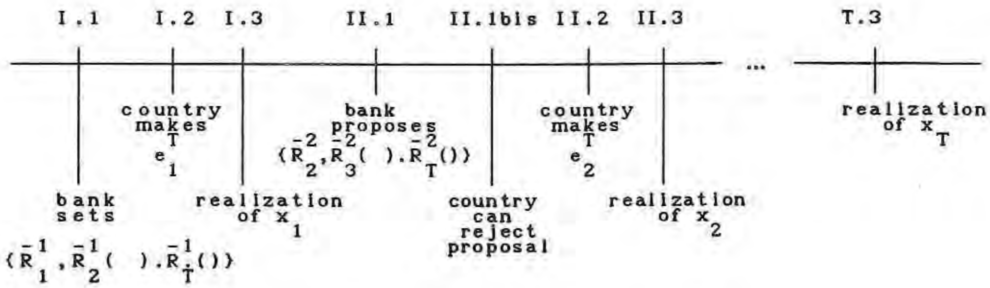
Let $\bar{R}_q^p(\cdot)$ $(p \leq q)$ be a function from $\mathbb{R}^{(q-p)}$ into $\mathbb{R}$ which assigns to each vector of payments received by the bank in periods p, p + 1, ...q − 1, an upper bound for the period q. In 1.1 the bank sets a vector $(\bar{R}_1^1, \bar{R}_2^1(\cdot), \ldots, \bar{R}_T^1(\cdot))$, where $\bar{R}_1^1 \in \mathbb{R}$ is the initial bound and $\bar{R}_q^1(\cdot)$, q = 1, 2,... T, assigns an upper bound for the q-th period as a function of all the previous repayments. In 1.2, knowing the legally binding announcement made by the bank, the country makes an effort $e_1^T \in \mathbb{R}_+$, which acts as a parameter in the distribution of x_1 which is realized in 1.3. At this date the bank is paid $R_1 = \min \{ \bar{R}_1^1, x_1 \}$ and the first period ends. We now describe the actions taken in period p, for $2 \leq p \leq T$. In p.1 the bank *proposes* a vector of upper bounds $(\bar{R}_p^p, \bar{R}_{p+1}^p(\cdot), \bar{R}_T^p(\cdot))$. In p.1 bis the country decides which vector will have legal validity. The set it can choose from consists of the proposal made by the bank in p.1 and the vector the country itself chose[1] in (p − 1).1 bis. If its choice is the former element, we say that it makes $A_p = 1$. Otherwise it makes $A_p = 0$. In p.2, knowing the bounds which have legal validity, the country makes an effort $e_p^T \in \mathbb{R}_+$. In p.3 nature chooses x_p according to (1.T) and the bank is repaid R_p equal to the minimum of x_p and the bound valid in period p.

The timing of the game is as the following figure shows.

---

[1] Except for p = 2, in which case the second element is the vector set in 1.1

Figure 4



Preferences of the country and the bank can be represented by the following VNM utility functions.

$$\text{country} \qquad \sum_{t=1}^{T} b^{t-1} H(x_t - R_t)$$

$$\text{bank} \qquad \sum_{t=1}^{T} b^{t-1} G(R_t)$$

<u>Equilibrium concept</u>

Let $Th_{1.2} = (\bar{R}^1_1, \bar{R}^1_2(\ ),\ldots \bar{R}^1_T(\ ))$, $Th_{1.3} = (Th_{1.2}, e^T_1)$ be the observed histories at 1.2 and 1.3, respectively.

For $2 \leq p \leq T$, let $Th_{p.1bis} = (Th_{(p-1).3}, x_{p-1}, R_{p-1})$, $Th_{p.2} = (Th_{p.1bis}, A_p)$, $Th_{p.3} = (Th_{p.2}, e^T_p)$

be the observed histories at p.1 bis, p.2 and p.3, respectively

A strategy for the country specifies, for the first period, a function $e^T_1(\cdot)$ which assign to each history $Th_{1.2}$ an effort level for the first period and, for period p, $2 \leq p \leq T$, a function $A_p(\cdot)$ which assigns to each history $Th_{p.1bis}$ an acceptance ($A_p(\cdot) = 1$) or rejection ($A_p(\cdot) = 0$) decision to the previous bank's proposal, and a function $e^T_p(\cdot)$ which assigns to each history $Th_{p.2}$, an effort level for the p-th period. We denote a country's strategy as

$$\mathbf{Te} = (e^T_1(\cdot),\ldots A_p(\cdot), e^T_p(\cdot),\ldots e^T_T(\cdot))$$

A strategy for the bank specifies, for the first period, a vector

$(\bar{R}^1_1, \bar{R}^1_2(\ ),\ldots \bar{R}^1_T(\ ))$ and, for each period p, $2 \leq p \leq T$, a function that assigns to each history $Th_{p.1}$ a vector $(\bar{R}^p_p, \bar{R}^p_{p+1}(\cdot), \bar{R}^p_T(\cdot))$.
We denote a bank's strategy as

$$\mathbf{TR} = ((\bar{R}^1_1, \bar{R}^1_2(\cdot),\ldots \bar{R}^1_T(\cdot)), \ldots(\bar{R}^p_p, \bar{R}^p_{p+1}(\cdot), \bar{R}^p_T(\cdot))(\cdot), \ldots\bar{R}^T_T(\cdot))$$

Define $TB_{t.r}(\ \mathbf{TR},\mathbf{Te},\ Th_{t.r}\ )$ and $TC_{t.r}(\ \mathbf{TR},\mathbf{Te},Th_{t.r}\ )$ as the expected utilities obtained by the bank and the country, respectively, evaluated at date t.r, when the history $Th_{t.r}$ has been observed and from that moment on the strategies **TR** ,**Te** are followed.

An optimal self-enforcing contract for the T-period model (OSECT) is a pair of strategies $\mathbf{TR}^*,\mathbf{Te}^*$, such that:

a) $\mathbf{TR}^* \in \underset{\mathbf{TR}}{\text{argmax}}\ TB_{t.r}(\mathbf{TR}, \mathbf{Te}, Th_{t.r}\ )$ $\forall$t.r in which the bank acts, $\forall\ Th_{t.r}$
b) $\mathbf{Te}^* \in \underset{\mathbf{Te}}{\text{argmax}}\ TC_{t.r}(\ \mathbf{TR}\ ,\mathbf{Te},\ Th_{t.r}\ )$ $\forall$ t.r in which the country acts, $\forall\ Th_{t.r}$

Proposition 11

There exists $\bar{b} \in (0,1)$ *s.t. if* $b \leq \bar{b}$ *, then in the T-period model,* $\forall\ T \in N$ *there exists an OSECT such that:*

a) the optimal strategy for the bank can be written as a set of one-period maturity debts $\{D_1^{T*}, D_2^{T*}, , D_T^{T*}\}$*, which are applied under the following rules:*

a.1) In the first period a debt $D_1^{T*} < \bar{x}\,(1-b^T)\,/\,(1-b)$ *is set*

a.2) If in t $\bar{x}$ *occurs for the first time, then in (t+1) a debt* $[D_t^{T*} - \bar{x}]/b$ *is set, which is repaid (in present value) in its entirety with probability one (the bank does not reduces the debt anymore)*

a.3) If in t, $t \geq 2$*,* $\bar{x}$ *has not occurred (* $x_m = \underline{x}\ \forall\ m < t$ *) then a debt* $D_t^{T*} < [D_{t-1}^{T*} - \underline{x}]/b$ *is set (it occurs a further reduction in relation to the unpaid amount of* D_{t-1}^{T*}*)*

b) The equilibrium efforts satisfy:

b.1) If in t $\bar{x}$ *occurs for the first time, then* $e_{t+h}^{T} = (\bar{x} - \underline{x})\quad \forall h \geq 1$

b.2) If in t, $t \geq 2$*, it has not occurred* $\bar{x}$ *(* $x_m = \underline{x}\ \forall\ m < t$ *), then* $e_t^T < e_{t-1}^T < (\bar{x} - \underline{x})$

b.3) The first period equilibrium effort, e_1^T*, satisfies:*
$\lim_{T \to \infty} e_1^T = \bar{e}\,(b, (\bar{x}-\underline{x}))$ *with* $\partial\bar{e}/\partial b > 0$*,* $(\bar{x}-\underline{x})/\,2 < \bar{e} < \gamma\,(\bar{x}-\underline{x})$*, for some* $\gamma \in (1/2,\ 1)$

APPENDIX 1. PROOFS OMITTED IN THE MAIN TEXT

Proof of proposition 2

We proceed by backward induction. *We* first establish the

Lemma 2.1

Under each OSEC2 the expected utility for the country in the last period, C_2, is greater or equal to $(\bar{x} - M)^2 / 2$. The expected utility for the bank in this period can be expressed as a function $V_2(\cdot)$ of C_2. Furthermore, $V_2(\cdot)$ is strictly decreasing, with first derivative continuous and (except in $(\bar{x} - x)^2/ 2)$ twice differentiable. Its derivatives are:

$$V_2'(C_2) = \begin{cases} 0 & \text{if } (\bar{x}-M)^2/ 2 = C_2 \\ (\bar{x}-x)/ (2C_2)^{1/2}-2 \in (-1,0) & \text{if } (\bar{x}-M)^2/ 2 < C_2 < (\bar{x}-x)^2/ 2 \\ -1 & \text{if } (\bar{x}-x)^2/ 2 < C_2 \end{cases}$$

$$V_2''(C_2) \begin{cases} < 0 & \text{if } \quad (\bar{x}-M)^2/ 2 < C_2 < (\bar{x}-x)^2/ 2 \\ = 0 & \text{if } \quad (\bar{x} -x)^2/ 2 < C_2 \end{cases}$$

Proof of Lemma 2.1

The country faces in II.2, when choosing e_2 , a one period problem with $\bar{R} = \bar{R}_2^2$. Since utility is additively separable in time, its decision does not depend on period one's events, that is, C_2 (**R**, **e**, h_1) depends only on e_2 and $\bar{R}_2^2$. So its optimal choice is given by (2) with $\bar{R}_2^2$ as its argument.

The same can be said for the bank: in II.1 it faces the same decision as in a one period model, with $\bar{R} = \bar{R}_2^2$. Then it follows from proposition 1 that if R_1 is such that $\bar{R}_2^1(R_1) > M$, then $\bar{R}_2^2() = M$ and, since the function $B\ (e^*(\bar{R}),\ \bar{R})$ is increasing in $\bar{R}$ for $\bar{R} \leq M$ it follows that $\bar{R}_2^2(\cdot) = \bar{R}_2^1(R_1)$ for those R_1 such that $\bar{R}_2^1\ (R_1) \leq M$.

Thus, in all OSEC2:

$\bar{R}_2^2\ (R_1)= \min\ \{\ M\ ,\ \bar{R}_2^1\ (R_1)\}\quad \forall R_1$

Now,

(i) If we take the function $C(e,\bar{R})$ used in the proof of proposition (1) and use the optimal strategy $e^*(\bar{R})$ defined in (2) in its first argument, we obtain the expected utility for the country as a function of only $\bar{R}$. Using $\bar{R} = \bar{R}_2^2$ we get:

$$C_2(\bar{R}_2^2) = \begin{cases} \underline{x} + (\bar{x} - \underline{x})^2/\ 2 - \bar{R}_2^2 & \text{if } \bar{R}_2^2 < \underline{x} \\ (\bar{x} - \bar{R}_2^2)^2/\ 2 & \text{if } \underline{x} < \bar{R}_2^2 \leq M \end{cases}$$

This function is one-to-one (strictly decreasing) and therefore has inverse.
$\bar{R}_2^2 \leq M$ implies that $C_2 \geq (\bar{x}\text{-}M)^2/\ 2$.
So, $\bar{R}_2^2$ can be written as a (strictly decreasing) function of the country's last period expected utility:

$$\bar{R}_2^2 = \begin{cases} \bar{x} - (2\ C_2)^{1/2} & \text{if } C_2 \leq (\bar{x} - \underline{x})^2/\ 2 \\ \underline{x} + (\bar{x} - \underline{x})^2/\ 2 - C_2 & \text{if } (\bar{x} - \underline{x})^2/\ 2 < C_2 \end{cases} \tag{4}$$

(ii) The expected utility for the bank at the beginning of the second period can be written as a function of only $\bar{R}_2^2$: If in (3) we use $\bar{R} = \bar{R}_2^2$ we get

$$B_2(\cdot) = \begin{cases} \bar{R}_2^2 & \text{if } \bar{R}_2^2 < \underline{x} \\ \underline{x} + (\bar{x} - \bar{R}_2^2)\ (\bar{R}_2^2 - \underline{x}) & \text{if } \underline{x} \leq \bar{R}_2^2 \leq M \end{cases} \tag{5}$$

We can then build $V_2(\cdot)$ through the composition of functions (4) and (5) :

$$V_2 = \begin{cases} \underline{x} + (2C_2)^{1/2}[\bar{x} - \underline{x} - (2C_2)^{1/2}] & \text{if } (\bar{x} - M)^2/\ 2 \leq C_2 \leq (\bar{x} - \underline{x})^2/\ 2 \\ \underline{x} + (\bar{x} - \underline{x})^2/\ 2 - C_2 & \text{if } (\bar{x} - \underline{x})^2/\ 2 \leq C_2 \end{cases}$$

from where the statements about $V_2'(\cdot)$ y $V_2''(\cdot)$ follow.

■ (Lemma 2.1)

The country's expected utility in I.2 is:

$$
C_1(\bar{R}^2_2\;, e_1, h_0) = \begin{cases} \underline{x} - \bar{R}^1_1 + b\, C_2(\bar{R}^2_2(\underline{x})\;) + \\ e_1\{\bar{x} - \bar{R}^1_1 + b\, C_2(\bar{R}^2_2(\bar{R}^1_1)) - b\, C_2(\bar{R}^2_2(\underline{x}))\} - e_1^2/\,2 & \text{if } \bar{R}^1_1 < \underline{x} \\ \\ b\, C_2(\bar{R}^2_2(\underline{x})\;) + \\ e\,\{\bar{x} - \bar{R}^1_1 + b\, C_2(\bar{R}^2_2(\bar{R}^1_1)) - b\, C\,(\bar{R}^2_1(\underline{x}))\} - e_1^2/\,2 & \text{if } \underline{x} \leq \bar{R}^1 \end{cases}
$$

Notice that $\bar{R}^2_2(\cdot)$ influences it only through $C_2(\cdot)$ and so can be written as a function of the utilities $C_2(\bar{R}^2_2(\underline{x}))$, $C_2(\bar{R}^2_2(\bar{R}_1))$ ignoring $\bar{R}^2_2(\bar{R}_1)$ and $\bar{R}^2_2(\underline{x})$. For notational convenience let us denote these utilities as $\underline{c}$ and $\bar{c}$, respectively. Then e_1^{2*} can be written as a function of $\underline{c}$, $\bar{c}$, and $\bar{R}^1_1$

$$
e_1^{2*} = \begin{cases} 0 & \text{if } \quad \bar{x} - \max\{\,\underline{x},\ \bar{R}^1_1\,\} \quad b\,(\bar{c} \quad \underline{c}) < 0 \\ \bar{x} - \max\{\,\underline{x},\ \bar{R}^1_1\,\} + b\,(\bar{c} - \underline{c}) & \text{if } 0 \leq \bar{x} - \max\{\,\underline{x},\ \bar{R}^1_1\,\} + b\,(\bar{c} - \underline{c}) \leq 1 \\ 1 & \text{if } 1 < \bar{x} - \max\{\,\underline{x},\ \bar{R}^1\,\} \quad b\,(\bar{c} \quad \underline{c}) \end{cases} \quad (6)
$$

The expected utility for the bank in I.1 is:

$$
B\;(\cdot) = \begin{cases} \bar{R}^1_1 + b\, B_2(\bar{R}_2(\underline{x}) + \; e_1^*\{\,\bar{x} - \underline{x} + b\, B_2(\bar{R}_2(\bar{R}_1) - b\, B_2(\bar{R}_2(\underline{x}))\} & \text{if } \bar{R}^1_1 < \underline{x} \\ \underline{x} + b\, B_2(\bar{R}_2(\underline{x})) + e_1^*\{\bar{R}_1 - \underline{x} + b\, B_2(\bar{R}_2(\bar{R}_1) - b\, B_2(\bar{R}_2(\underline{x}))\} & \text{if } \underline{x} \leq \bar{R}^1_1 \end{cases}
$$

So, by lemma (2.1), the bank's expected utility can be expressed as a function of $\bar{c}$, $\underline{c}$, and $\bar{R}^1_1$, and equals:

$$
\Pi(\bar{R}^1_1, \underline{c}\;, \bar{c}) = \begin{cases} \bar{R}^1_1 + bV_2(\;\underline{c}\;) + e_1^{2*}\;\{\;\bar{x} - \underline{x} + b\;V_2(\;\bar{c}\;) - b\;V_2(\;\underline{c}\;)\} & \text{if } \bar{R}^1_1 < \underline{x} \\ \\ \underline{x} + bV_2(\;\underline{c}\;) + e_1^{2*}\;\{\;\bar{R}^1_1 - \underline{x} + b\;V_2(\;\bar{c}\;) - b\;V_2(\;\underline{c}\;)\;\} & \text{if } \underline{x} \leq \bar{R}^1_1 \end{cases}
$$

with e_1^{2*} defined as in (6)
So, the bank's problem in I.1 is

$$\underset{\bar{R}_1^1, \underline{c}, \bar{c}}{\text{Max}} \quad \Pi\,(\bar{R}_1^1, \underline{c}, \bar{c}) \quad \text{s.a. } \underline{c}, \bar{c} \geq (\bar{x} - M)^2 / 2 \qquad (7)$$

It is straightforward that $\bar{R}_1^1 < \underline{x}$ is not optimal because

$\bar{R}_1^1 < \underline{x}$ implies $\partial \Pi / \partial \bar{R}_1^1 = 1 > 0$.

Consider then the region $\bar{R}_1^1 \geq \underline{x}$

We know from (6) that we need to consider separately each of the different regions for e_1^{2*}.

Consider the

CASE $\bar{x} - \bar{R}_1^1 + b\,(\bar{c} - \underline{c}) \in (0,1)$

In this case e_1^{2*} is differentiable.

Note first that a necessary condition for an optimum is

FACT (1) $\{ \bar{R}_1^1 - \underline{x} + b\, V_2(\bar{c}) - b\, V_2(\underline{c}) \} \equiv A > 0$, since:

Suppose $A < 0$. Then:

$\partial \Pi / \partial \bar{R}_1 = e_1^* - A > 0$

$\partial \Pi / \partial \bar{c} = e_1 b\, V_2'(\bar{c}) + b\, A < 0$ if $\bar{c} > (\bar{x} - M)^2 / 2$

Thus, the bank's optimization leads us to

$\bar{R}_1^1 = \bar{x}$, $\bar{c} = (\bar{x} - M)^2 / 2$, but then

$A = \{ \bar{x} - \underline{x} + V_2((\bar{x} - M)^2 / 2) - V_2(\underline{c}) \} \geq (\bar{x} - \underline{x}) > 0$!!

We can now see that

FACT (2) $\underline{c} = (\bar{x} - M)^2 / 2$.

Assume this equality does not hold. Then

$\partial \Pi / \partial \underline{c} = b\, (1 - e_1^*) V_2'(\underline{c}) - b\, A < 0$

(because $V_2'(\underline{c}) < 0$) which is a contradiction.

We now establish the

Lemma 2.2

(i) If the pair $\{ \bar{R}_1^1, \bar{c} \}$ solves (7) with $\bar{R}_1^1 < \bar{x}$, then $\bar{c} \geq (\bar{x} - \underline{x})^2 / 2$

(ii) There exists a pair $\{ \bar{R}_1^1, \bar{c} \}$ with $\bar{R}_1^1 = \bar{x}$ which solves (7)

Proof of Lemma 2.2

(i) Assume $\bar{R}^1_1 < \bar{x}$ and $\bar{c} < (\bar{x} - \underline{x})^2/ 2$

Consider the bank's problem as the maximization of Π with $\underline{c} = (\bar{x}-M)^2/ 2$ and a fixed e_1^{2*} value, so that (6) is now a restriction, and let λ be the associated multiplier. Note that if $\bar{R}^1_1 < \bar{x}$ and $\underline{c} < (\bar{x}-\underline{x})^2/ 2$ the objective function is differentiable in both variables. Deriving the lagrangean with respect to $\underline{c}$ y $\bar{R}^1_1$ we obtain the first order conditions:

$$e_1 V'_2(\bar{c})) = -\lambda \text{ , and}$$

$$e_1 = \lambda,$$

respectively. But $-V'_2(\bar{c}) < 1$ if $\bar{c} < (\bar{x}-\underline{x})^2/ 2$, which is a contradiction.

So, if we set $\bar{c} < (\bar{x}-M)^2/ 2$ and $R^1_1 < \bar{x}$ it is feasible for the bank to, through an increase in both variables, induce the same e^*_1 and yet increase its earnings under $\bar{x}$, because when increasing $\bar{c}$, the amount given up by the bank is lower than the additional gain to the country: there will be a lower overhang in the future. In contrast, when increasing $\bar{R}^1_1$ a zero sum reallocation occurs. (This ends the proof of lemma 2-(i))

(ii) Assume the pair $\{ \bar{R}^1_1, \bar{c} \}$ with $\bar{R}^1_1 < \bar{x}$ is optimal.

Then it must be the case that $\bar{c} > (\bar{x}-\underline{x})^2/ 2$

Now, $y > (\bar{x}-\underline{x})^2/ 2$ implies $V'_2(y) = 1$, and so if we increase $\bar{c}$ and $\bar{R}^1_1$ they can be treated as perfect complements. (This ends the proof of lemma 2-(ii))

■ (Lemma 2.2)

Consider now the problem (7) with $\underline{c} = (\bar{x}-M)^2/ 2$ and $\bar{R}^1_1 = \bar{x}$.
Then we have

$$\underset{\bar{c}}{\text{Max}} \quad \underline{x} + b\, V_2(\underline{c}) + b \{ \bar{c} - \underline{c} \} \{ \bar{x} - \underline{x} + b\, V_2(\bar{c})) - b\, V_2(\underline{c}) \} \tag{8}$$

$$\text{s.t. } \bar{c} \geq (\bar{x} - M)^2 / 2$$

To solve this problem build the auxiliary function

$$V^A_2(c) = \underline{x} + (\bar{x}-\underline{x})^2/ 2 - c$$

Note that if in the objective function we write $V^A_2(\cdot)$ instead of $V_2(\cdot)$, the resulting function is equal to the original one if $\bar{c} \geq (\bar{x}-\underline{x})^2/\ 2$ and higher otherwise. The problem (8) so transformed is the maximization of a concave function (the second derivative of the objective function with respect to $\bar{c}$ is $2\ b^2\ V^{A}\ '(\bar{c}) < 0$) and the first order condition is:

$$\bar{c} = \{\ \bar{x}-\underline{x} + b\ [\underline{x} + (\bar{x}-\underline{x})^2/\ 2] - b\ V_2((\bar{x}-M)^2/\ 2) - b(\bar{x}-M)^2/\ 2\}/\ 2b + (\bar{x}-M)^2/\ 2 \tag{9}$$

Since $\bar{c}^* > (\bar{x}-\underline{x})^2/\ 2$, then it also solves the original problem (8) .

We have been considering the case $0 < (\bar{x} - \bar{R}^1_1) + b\ (\bar{c} - \underline{c}) < 1$. We will next see that outside this region the objective function takes on values which are below the value just found.

CASE $(\bar{x} - \bar{R}^1_1) + b\ (\bar{c} - \underline{c}) \leq 0$

The solution to the country's problem is $e_1 = 0$
The value of the objective function is then:
$\underline{x} + b\ V\ (\underline{c}) \leq \underline{x} + b\ V\ (c^{min}_0)$
and the right hand side of this inequality is the value of the objective function in problem (8) when $\bar{c} = c^{min}_0$, which we Know it is not optimal.

CASE $(\bar{x} - \bar{R}^1_1) + b\ (\bar{c} - \underline{c}) \geq 1$

The solution to the country's problem is $e_1 = 1$
Maximize now the bank's objective function under this situation, that is, solve
Max $\bar{R}^1_1 + b\ V\ (\bar{c})$
s.t. $(\bar{x}- \bar{R}^1_1) + b\ (\bar{c} - \underline{c}) \geq 1$
Notice that:
i) $\underline{c} = c^{min}_0$ is not optimal (it is straightforward)
ii) The objective function evaluated in any pair $(\bar{R}^1_1\ ,\ \bar{c})$ is no higher than evaluated in $(\bar{x}\ ,\ \bar{c})$. Let us work with $\bar{R}^1_1 = \bar{x}$.
Then we have the problem:
Max $\bar{x} + b\ V(\bar{c})$
$\bar{c}$ s.t. $b\ (\bar{c} - c^{min}_0) \geq 1$
Since $V\ (\bar{c})$ is decreasing, the value of $\bar{c}$ which solves this problem is such

that the restriction is satisfied with equality. Then the value of the objective function is

$\bar{x} + b\, V(1/\, b + c_0^{min}) \leq \bar{x} + b\, V^A(1/b + c_0^{min})$

but the right-hand side is just the objective function in (8) when

$\bar{c} = 1\,/\,b + c_0^{min}$ and we know this value of $\bar{c}$ is not optimal.

Thus, we have found that $\bar{R}_1^1 = \bar{x}$, $\bar{R}_2^1(\underline{x}) = M$, $\bar{R}_2^2(x_1) = \bar{R}_2^1(x_1)$ $\forall\ x_1$, and $\bar{R}_2(\bar{R}_1)$ set as to satisfy (9) is an optimal strategy for the bank. Nevertheless, this solution is not unique. Though sufficient, it is not necessary for an optimal contract. It allows the following variations.

If in (8) we write $\bar{R}_1^1$ instead of $\bar{x}$ the resulting first order condition is

$$\bar{R}_1^1 + b\, \bar{R}_2^1(\bar{R}_1^1) = b\,(\underline{x} + 5\,(\bar{x}-M)^2/4) + M < b\,\underline{x} + \bar{x}$$

Set $\bar{R}_1^1 = \bar{x}$ leads to $\bar{R}_2^1(\bar{R}_1^1) < \underline{x}$. By lemma 2.2 we know $\bar{R}_1^1 < \bar{x}$ is part of an optimal contract only if $\bar{c} > (\bar{x}-\underline{x})^2/\ 2$, that is,

$\bar{R}_1^1 < \bar{x} \Rightarrow \bar{R}_2^1(\bar{R}_1^1) \leq \underline{x}$

We can then choose $\{\ \bar{R}_1^1\ ,\ \bar{R}_2^1(\bar{R}_1^1)\ \}$ with $\bar{R}_1^1 < \bar{x}$, as long as it satisfies (ii) and $\bar{R}_2^1(\bar{R}_1^1) \leq \underline{x}$. This is the first variation.

The second one is that $\underline{c} = (\bar{x} - M)^2/\ 2$ is equivalent to any pair $(\bar{R}_2^1(\underline{x}),\ \bar{R}_2^2(h_{II.1})$ satisfying $\bar{R}_2^1(\underline{x}) > M$ and $\bar{R}_2^2\ (h_{II.1}) = M$ if $x_1 = \underline{x}$

Notice that neither agent's expected utilities are altered by any of the previous variations. Finally, to see why (i) is true, notice that the derivative of the objective function evaluated at $\underline{c}^*$ is:

$b\ [\ 1 - e_1^*\ ]V_2'(\underline{c}^*)\ - b\,A = 0 - b\,A < 0$

i.e., if we decrease the country's expected utility in the second period if in the first period $\underline{x}$ occurs, this has a second order effect in the bank's expected profits (since the highest credible punishment is one in which the bank maximizes its profits at the beginning of the second period) but it has a first order effect in the incentives it gives to the country to make effort.

∎ (Proposition 2)

Proof of proposition 7

Note first that given a pair of bounds $\bar{R}^{\gg}_1$, $\bar{R}^{\gg}_2(R_1)$, there is uniqueness in the country's expected utility, for any continuation OSEC with such initial bounds.

In I.1 bis the country will face a proposal $\bar{R}^{\gg}_1$, $\bar{R}^{\gg}_2(R_1)$. Since it can only give an answer of acceptance or rejection to that proposal, it will accept if and only if country's expected utility under $\bar{R}^{\gg}_1$, $\bar{R}^{\gg}_2(R_1) \geq$ country's expected utility under $\bar{R}^{IN}_1$, $\bar{R}^{IN}_2(R_1)$

So the bank's problem in I.1 is:

$$\max_{\bar{R},\ \bar{c},\ \underline{c}} \quad \underline{x} + V(\underline{c}) + e\,[V(\bar{c}) - V(\underline{c}) + \bar{R} - \underline{x}]$$

s.t.

$$e \in \operatorname{argmax}\ \underline{c} + \hat{e}\,[\bar{x} - \bar{R} + \bar{c} - \underline{c}] - \hat{e}^2/2 \qquad \text{(IC)}$$

$$\hat{e} \in [0,1]$$

$$\underline{c} + e\,[\bar{x} - \bar{R} + \bar{c} - \underline{c}] - e^2/2 \geq c_1^{cont} \qquad \text{(VP)}$$

$$\underline{c}\ ,\ \bar{c} \geq c_0^{min}$$

Due to the concavity of the country's objective function, we can substitute IC by the first order condition

$$e_1 = \begin{cases} 0 & \text{si } \bar{x} - \bar{R} + \bar{c} - \underline{c} < 0 \\ \bar{x} - \bar{R} + \bar{c} - \underline{c} & \text{si } \bar{x} - \bar{R} + \bar{c} - \underline{c} \in [0,1] \\ 1 & \text{si } \bar{x} - \bar{R} + \bar{c} - \underline{c} > 1 \end{cases}$$

We will make use of the following definitions:

Let

c_t^{min} be the country's expected utility under every OSEC with infinite initial bounds in the t period model,

c_t^{max} the country's expected utility when it has to pay $\underline{x}$ each period in the t period model, and

$c_0^{one}(c_1) = c_1 - 1/2$.

We can get rid of the first period bound due to the following fact

Fact 7.1 For each triplet ($\bar{R}$, $\bar{c}$,$\underline{c}$) there exists another triplet ($\bar{x}$, $\bar{c}'$, $\underline{c}$) which provides the country with the same expected utility and the bank with at least the same expected utility. If it provides the bank with the same expected utility then it induces the same first period effort from the country.

Proof

Let $c_0^{max} = (\bar{x} - \underline{x})^2/ 2$ (the country's expected utility in a one period model with debt equal to $\underline{x}$).

If $\bar{c} \geq c_0^{max}$, the bank's expected utility remains the same because $V'(y) = -1 \;\; \forall \;\; y \geq \bar{c}$.

From the country's problem first order condition it follows that the first period effort is also the same.

If $\bar{c} < c_0^{max}$, then the bank's expected utility is strictly greater because $|V'(y)| < 1$ in a neighborhood of $\bar{c}$

□ (Fact 7.1)

We will use $\bar{R} = \bar{x}$ and so we have a problem in two variables: $\bar{c}$ and $\underline{c}$. We can also establish

Fact 7.2 Choosing $\underline{c} \geq \bar{c}$ is not optimal.

Proof.

If the bank did it, it would lead to the following expected utilities

for the bank = $\underline{x} + V(\underline{c})$, and for the country = $\underline{c}$

If, instead, we take $\bar{c}'$ such that $\bar{c} < \bar{c}'$ and $V(\underline{c}) - V(\bar{c}') < (\bar{x} - \underline{x})$ (which we can do thanks to the continuity of $V(\cdot)$), we will have

$$\text{bank's EU} = \underline{x} + V(\underline{c}) + [\bar{c}' - \underline{c}]\,[V(\bar{c}') - V(\underline{c}) + \bar{x} - \underline{x}] >$$

$$\underline{x} + V(\underline{c})$$

and country's EU = $\underline{c} + [\bar{c}' - \underline{c}]^2/ 2 > \underline{c}$.

□ (Fact 7.2)

Remember we have defined $c_1 = \max \{ c_1^{min}, c_1^{cont} \}$. If $c_1 = c_1^{min}$ then we face the infinite initial debt problem already studied: the restriction VP is satisfied by the solution to the unrestricted problem. The presence of the initial bounds is in this case irrelevant.

Consider the case $c_1 = c_1^{cont} > c_1^{min}$. Changes in $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$ can cause changes in c_1, i.e., c_1 is a non-constant function of $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$. To emphasize this dependence we will often write $c_1(\bar{R}_1^{IN}, \bar{R}_2^{IN}(R_1))$

It only remains to make sure that in this case, when the initial bounds are relevant, the bank does not allow the country to obtain more expected utility than that provided by the initial bounds.

Fact 7.3. If $c_1 = c_1^{cont} > c_1^{min}$ the restriction VP is satisfied with equality.

The reason is that $\forall\ c_1 > c_1^{min}$ in a neighborhood of c_1 the bank can take advantage of any decrease in this variable to increase its expected utility. To see it, consider a value of c_1 which comes out of $\{ \bar{R}_t^t, \bar{c}, \underline{c} \}$ $\bar{R}_t^t = \bar{x}$ and $\bar{c} \geq \underline{c}$.

i) if $\underline{c} > c_2^{min}$ we can choose

$\underline{c}' = \underline{c} - h$

$\bar{c}' = \bar{c} - h$,

with $0 < h < \underline{c} - c_2^{min}$, so that

e is constant

$V_2(\underline{c}') > V_2(\underline{c})$ and $V_2(\bar{c}') > V_2(\bar{c})$

because $V_2(\cdot)$ is strictly decreasing.

As a result, the bank's expected utility under $(\underline{c}', \bar{c}')$ is strictly greater than under $(\underline{c}, \bar{c})$.

ii) if $\underline{c} = c_2^{min}$, then we are in the situation analyzed in the infinite initial debt case. By the solution to problem (8) in the proof of proposition 2, we know that the expected utility of the bank reaches its maximum at $\bar{c}^*$ such that $C_{1.2}(c_2^{min}, \bar{c}^*, \bar{x}) = c_1^{min}$ and is strictly concave in $\bar{c}$. Then it is strictly decreasing in $\bar{c}$ for $\bar{c} \geq \bar{c}^*$.

Then if we decrease $\bar{c}$ leaving unchanged $\underline{c}$ and $\bar{R}_1^1$, we obtain a decrease in c and an increase in the bank's expected utility.

□ (Fact 7.3)

Define the function

$$\bar{c} : \{ (\underline{c}, c_1) \in R^2 : \underline{c} \leq c_1 \} \longrightarrow R$$
$$(\underline{c}, c_1) \longrightarrow \bar{c}(\underline{c}, c_1)$$

such that

$$\underline{c} + [\bar{c} - \underline{c}]^2 / 2 = c_1 \text{ and } \bar{c} \geq \underline{c}$$

Notice that $\bar{c}(\cdot,\cdot)$ is a continuous function and, in the interior of its domain, differentiable.

The fact that the 'voluntary participation' restriction is satisfied with equality for $c_1 \geq c_1^{min}$ (and if $c_1 < c_1^{min}$ we already know the solution to the problem) will allow us to transform the initial bank's problem into the maximization of a continuous function (in one variable) in a compact set, as we establish in the following lemma

Lemma 7.1 Under every OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$, at date I.1 over a two period horizon, the expected utility for the country is equal to $c_1(\bar{R}_1^{IN}, \bar{R}_2^{IN}(R_1))$, and if $c_1 \leq c_1^{max}$, the expected utility for the bank is equal to the value of the following

(Program *)

$$\operatorname*{Max}_{\underline{c}} P(\underline{c},c_1) = \underline{x} + V(\underline{c}) + [\bar{c}(\underline{c};c_1) - \underline{c}]\,[V(\bar{c}(\underline{c};c_1)) - V(\underline{c}) + \bar{x} - \underline{x}]$$

$$\text{s. a.} \qquad \underline{c} \in [\max\{c_0^{min}, c_0^{one}(c_1)\}, \min\{c_1, c_0^{max}\}]$$

Furthermore, if the value of $\underline{c}$ which solves this program is unique and equal to $\underline{c}^*$, then under every OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$, if $x_1 = \underline{x}$ then in II.1 the country's expected utility is $\underline{c}^*$ an the bank's expected utility is $V(\underline{c}^*)$.

Proof of lemma 7.1

It only remains to justify the restrictions $\underline{c} \geq c_0^{one}(c_1)$ and $\underline{c} \leq c_0^{max}$.

As for the first one, we have that:

i) For every triplet $(\bar{R}, \underline{c}, \bar{c})$, with $[\bar{x} - \bar{R} + \bar{c} - \underline{c}] \geq 1$ there exist another triplet $(\bar{R}, \underline{c}', \bar{c})$ with $[\bar{x} - \bar{R} + \bar{c} - \underline{c}'] = 1$ which provides the same expected utility to both agents because

i.i) $[\bar{x} - \bar{R} + \bar{c} - \underline{c}] \geq 1 \quad \Rightarrow \quad$ EU bank $= V(\bar{c}) + \bar{R}$ and

EU country $= \bar{x} - \bar{R} + \bar{c} - 1/2$

i.e., neither of both expected utilities depend on $\underline{c}$.

i.ii) notice that $\underline{c}' > \underline{c}$, so that if $\underline{c}$ is feasible ($\underline{c} > c_0^{min}$), $\underline{c}'$ will also be feasible.

Then, there is no loss in generality in restricting the bank to choose $\underline{c}$, $\bar{c}$ such that $\bar{c} - \underline{c} \leq 1$.

As for the second one:

ii) Assume we allow $\underline{c} \geq c_0^{max}$. Choosing $\underline{c} \geq c_0^{max}$ is optimal if and only if $c_1 \geq c_1^{max}$, because:

Assume $\underline{c} \geq c_0^{max}$ is optimal. Remember that, from fact 7.2, $\bar{c} > \underline{c}$.

Then $V'(\underline{c}) = V'(\bar{c}) = -1$, and we have, from the first order condition:

$V(\bar{c}(\underline{c}, c_1)) - V(\underline{c}) + \bar{x} - \underline{x} = 0$

Then $\bar{c} - \underline{c} = \bar{x} - \underline{x}$, from which

$c_1 \geq c_0^{max} + (\bar{x} - \underline{x})^2 / 2 \geq 2\, c_0^{max} = c_1^{max}$

Conversely, if $c_1 \geq c_1^{max}$, the bank can withdraw less than $\underline{x}$ each period, notwithstanding the realization of x_1, without causing any distortions in the country's effort, so that the country's expected utility in the last period if in the first one occurs $\underline{x}$ is at least $(\bar{x} - \underline{x})^2 / 2 = c_0^{max}$

(Since the solution to the problem when $c_1 > c_1^{max}$ is trivial, we will restrict our attention to the case $c_1 \leq c_1^{max}$. This will allow us to restrict our attention to the case $\underline{c} \leq c_0^{max}$)

■(Lemma 7.1)

And thus Weierstrass Theorem guarantees the existence of a global maximum.

■(Proposition 7)

Proof of proposition 8

We have to solve the program (*), the maximization of a function of one variable ($\underline{c}$).

The derivative of the objective function with respect to $\underline{c}$ is a function of $\underline{c}$, parameterized by c_1. Let's call it FD $(\underline{c}\,;\, c_1)$

Deriving the objective function and simplifying we obtain:

$FD(\underline{c}\,;\, c_1) = [1 - e(\underline{c}, c_1)]\,[\,V'(\underline{c}) - V'(\bar{c})\,] - A(\underline{c}\,; c_1) / e(\underline{c}, c_1)$

where

$A(\underline{c}\,; c_1) \equiv [\,V(\bar{c}(\underline{c}\,; c_1)) - V(\underline{c}) + \bar{x} - \underline{x}\,]$

and remember that

$e(\underline{c}\,;\, c_1) \equiv \bar{c}(\underline{c}\,;\, c_1) - \underline{c}$

For notational simplicity, in what remains of this proof we will omit the arguments of $A(\cdot)$ and $e(\cdot)$. These functions are continuous and differentiable and

$\partial A / \partial \underline{c} = - V'(\bar{c})(1-e) / e - V'(\underline{c}) > 0$ if $0 < e \leq 1$

$\partial A / \partial c_1 = V'(\bar{c}) / e < 0$ if $0 < e$

$\partial e / \partial \underline{c} = -1/e < 0$ if $0 < e$

$\partial e / \partial c_1 = 1/ e > 0$ if $0 < e$

Notice the following facts:

<u>Fact (8.1)</u> $\bar{c} \geq c_0^{max}$

<u>Proof</u> If at the optimum $\underline{c} = c_0^{min}$, then, since in the infinite debt problem the country obtains $c_1 = c_1^{min}$ through $\underline{c} = c_0^{min}$ and $\bar{c} > c_0^{max}$, if we want to provide the country with $c_1 > c_1^{min}$ we will also need $\bar{c} > c_0^{max}$

If we have an interior optimum,

(i) $e > A$ because it must be the case that

$FD(\underline{c}, c_1) = (1 - e)\ (V'(\underline{c}) - V'(\bar{c})) - A/e = 0$

from where $A/ e = (1-e)\ (V'(\underline{c}) - V'(\bar{c}))$

Now, $(V'(\underline{c}) - V'(\bar{c})) \in (0,1)$ because $V''(\ ) < 0$ and, thus

$0 < e \leq 1 \Rightarrow e > A$

ii) Now, $e + A \equiv E(\underline{c}) \geq E(c_0^{max}) = \bar{x} - \underline{x}$

From (i) and (ii) it follows that $e > (\bar{x} - \underline{x}) / 2$

Now, at the optimum, $e = \bar{c} - \underline{c}$, from where

$\bar{c} = \underline{c} + e \geq \underline{c} + (\bar{x}-\underline{x})/ 2 > c_0^{max}$

where the second inequality follows from

$c_0^{max} - c_0^{min} = (\bar{x} - \underline{x})^2/ 2 - (\bar{x} - M)^2/ 2 = 3\ (\bar{x} - M)^2/ 2$

and $3\ (\bar{x} - M)^2/ 2 > (\bar{x} - M)$ iff $(\bar{x} - M) < 2 / 3$, which is the case

□(Fact 8.1)

This result will allow us to work with the same auxiliary function $V^A(\cdot)$ we used in the previous section.

Let $E(\underline{c}) \equiv (\bar{x} - \underline{x}) + c_0^{max} + V(c_0^{max}) - \underline{c} - V(\underline{c})$.

It is useful to interpret $E(\underline{c})$ as the total gains from obtaining a 'good' instead a 'bad' state. Since at the optimum $\bar{c} > c_0^{max}$, such gains depend only on $\underline{c}$.

Notice that

$A(\underline{c}, c_1) + e(\underline{c}, c_1) = E(\underline{c})\ \forall\ \underline{c}, c_1$

and that $dE(\underline{c}) / d\underline{c} < 0$

Fact (8.2) $0 < e \leq 1$ and $A < 0 \Rightarrow FD(\underline{c}; c_1) > 0$

Proof $\bar{c} - \underline{c} > 0 \Rightarrow \bar{c} > \underline{c} \Rightarrow V'(\bar{c}) > V'(\underline{c})$

and since $(1 - e) > 0$, the first term is positive.

Since $A < 0$ and $e > 0$, then the second term is also positive.

□(Fact 8.2)

Fact 8.3 $\exists\, \epsilon > 0$ s.t. if $\underline{c} \geq \min\{c_1, c_0^{max}\} - \epsilon$, then $FD(\underline{c}; c_1) < 0$ (The value of $\underline{c}$ which solves program * is not close -neither at- to the upper frontier of the feasible set)

Proof

i) $\lim_{\underline{c} \to c_1} FD(\underline{c}; c_1) = -\infty$

Proof. As $\underline{c}$ approaches c_1, e approaches zero, and so:

the first term $(1-e)(V'(\underline{c}) - V'(\bar{c}))$ is bounded

and the second one$(-A/e)$ diverges to $-\infty$because A approaches a strictly

positive number $(E(c_1) \geq \bar{x} - \underline{x}$ because $e(\underline{c};c_1) + A(\underline{c};c_1) \equiv E(\underline{c})$ approaches $E(c_1))$

This proofs the fact if $c_1 \leq c_0^{max}$. (At $\underline{c} = c_1$, $P(\underline{c}, c_1)$ is not differentiable but it is clearly decreasing in $\underline{c}$)

ii) If $c_0^{max} < c_1 < c_1^{max}$ then $\lim_{\underline{c} \to c_0^{max}} FD(\underline{c}; c_1) < 0$

Proof:

$c_1 > c_0^{max} \Rightarrow \lim_{\underline{c} \to c_0^{max}} e > 0$ because $e \equiv [2(c_1 - \underline{c})]^{1/2}$

on the other hand:

$\lim_{\underline{c} \to c_0^{max}} A(\underline{c}, c_1) > 0 \quad \forall\, c_1 < c_1^{max}$, because:

$c_0^{max} + E(c_0^{max})^2/2 = c_0^{max} + (\bar{x} - \underline{x})^2/2 = c_0^{max} + c_0^{max} = c_1^{max}$, from where

$e(c_0^{max}, c_1^{max}) = E(c_0^{max})$

$A(c_0^{max}, c_1^{max}) = E(c_0^{max}) - e(c_0^{max}, c_1^{max}) = 0$

and $\partial A / \partial c_1 < 0$

and then $c_1 < c_1^{max} \Rightarrow A(c_0^{max}, c_1^{max}) > 0$

The statement then follows from the continuity of A.

Finally, notice that

$$\lim_{\underline{c} \to c_0^{max}} e > 0 \Rightarrow \lim_{\underline{c} \to c_0^{max}} V'(\underline{c}) - V'(\bar{c}(\underline{c}; c_1)) = 0$$

(The fact that $\underline{c} = c_0^{max}$ is not optimal has already been justified)

□(Fact 8.3)

Fact (8.4) if $\underline{c}$, c_1 are such that $0 < e < E(\underline{c}) \leq 1$ then

$\partial FD (\underline{c};c_1) / \partial c_1 > 0$

<u>Proof</u> $\partial FD(\underline{c},c_1) / \partial c_1 = \{ E(\underline{c}) - e^2 [1 + V'(\underline{c})] \} / e^3$

□(Fact 8.4)

The following lemma will be helpful in determining when it can be optimal to set $\underline{c} = c_0^{min}$ because it tells us that, for 'small' c_1, the derivative of the objective function in $\underline{c} = c_0^{min}$ is negative, and for 'big ' c_1 it is either positive or it produces $\bar{c} - \underline{c} > 1$, which is not feasible.

<u>Lemma 8.1</u> (Analysis of $FD(c_0^{min}, c_1)$ for different values of c_1)

If $E(c_0^{min}) < 1$, *then*

there exists c_1^* , *with* $c_1^{min} < c_1^* < c_0^{min} + E(c_0^{min})^2 < c_1^{max}$ *s. t.*[1]

(i) $c_1 < c_1^* \Rightarrow FOC\ (c_0^{min}, c_1) < 0$

(ii) $c_1^* < c_1 < c_0^{min} + E(\underline{c}_0^{min})^2 / 2 \Rightarrow FOC\ (c_0^{min}, c_1) > 0$

(iii) $c_0^{min} + E(c_0^{min})^2 / 2 < c_1 \Rightarrow FOC\ (c_0^{min}, c_1) > 0$ " $\bar{c} - \underline{c} > 1$

<u>Proof</u>

We will prove that $FD(c_0^{min}; c_1)$ is negative when $c_1 = c_1^{min}$, positive in $c_1 = c_0^{min} + E(\underline{c}_0^{min})^2 / 2$, and increasing in $c_1 \in [c_1^{min}, c_0^{min} + E(\underline{c}_0^{min})^2 / 2]$

a) $FD(c_0^{min}; c_1^{min}) < 0$

because a.1) $V'(c_0^{min}) = 0$

a.2) c_0^{min} ; c_1^{min} are the expected utilities the country gets in the infinite debt model in which we have $\bar{c}^* > c_0^{max}$ and so

$V'(\bar{c}^*) = -1$ and $A = e > 0$

Thus $FD(c_0^{min}; c_1^{min}) = (1 - e) - 1 = -e < 0$

b) $E(\underline{c}_0^{min}) < 1 \Rightarrow FD(\underline{c}_0^{min}; c_0^{min} + E(\underline{c}_0^{min})^2 / 2) > 0$

Proof

$FD(\underline{c}_0^{min}; c_0^{min} + E(\underline{c}_0^{min})^2 / 2) = (1 - E(\underline{c}_0^{min}))(-V'(\bar{c}))$

because $[A(\underline{c}_0^{min}; c_0^{min} + E(\underline{c}_0^{min})^2 / 2) = 0$

and $V_0'(\underline{c}_0^{min}) = 0$,

---

[1]To see that $c_0^{min} + E(c_0^{min})^2 < c_1^{max}$ notice that

$c_0^{min} + E(c_0^{min})^2 = \text{Max}\ (1-e)\, c_0^{min} + e(c_0^{min} + E(c_0^{min})) - e^2/2$, and

$c_1^{max} = c_0^{max} + c_0^{max} = \text{Max}(1-e) + e(c_0^{max} + (\bar{x} - \underline{x})) - e^2/2$

We have that $c_0^{min} < c_0^{max}$.

Furthermore, $c_0^{min} + E(c_0^{min})^2 < c_0^{max} + (\bar{x} - \underline{x})$ because this inequality is equivalent to $V(c_0^{max}) < V(c_0^{min})$

c) $c_1 < c_0^{min} + [\ E(\underline{c}_0^{min})^2 / 2 \quad \Rightarrow \quad \partial\, FD(\ \underline{c}_0^{min},\ c_1) / \partial\, c_1 > 0$

Proof

$c_1 < c_0^{min} + E(\underline{c}_0^{min})^2 / 2 \ \Rightarrow \ e < E(\underline{c}_0^{min}) \ \Rightarrow \ A > 0$

(because $e(\underline{c}_0^{min};c_1) + A(\underline{c}_0^{min};c_1) \equiv E(\underline{c}_0^{min};c_1) \quad \forall\, c_1 \geq c_1^{min})$, and thus we can apply fact 8.4)

Thus, (a), (b) and (c) imply points (i) y (ii) of the lemma because we have an strictly increasing function which in c_1^{min} is negative and in $[c_0^{min} + E(\underline{c}_0^{min})^2 / 2\ ]$ is positive, with $[c_0^{min} + E(\underline{c}_0^{min})^2 / 2\ ] > c_1^{min}$

Finally, (iii) follows from

$c_0^{min} + E(\underline{c}_0^{min})^2 / 2 < c_1 \Rightarrow E(\ \underline{c}_0^{min};c_1) < e\ (\underline{c}_0^{min};c_1)$

and

$A\ (\underline{c}_0^{min};c_1) \equiv E(\ \underline{c}_0^{min};c_1) - e(\underline{c}_0^{min};c_1) < 0 \qquad \forall\, c_1 \geq c_1^{min}$

and then we can apply fact 2.5.

■(Lemma 8.1)

Lemma 8.1 (2)

If $E(\ c_0^{min}\ ) > 1$, then $FOC(c_0^{min}\ , c_1) < 0 \quad \forall\, c_1$

Proof

$FD(c_0^{min}\ , c_1) = (1-e)\ (\ V'(c_0^{min}\ ) - V'(\ \bar{c}\ ) - (\ E - e) / e$

Using $V'(\ c_0^{min}) = 0$ and $V'(\ \bar{c}\ ) = 1$ we obtain

$FD(\underline{c},\ c_1) \geq 0$ iff $\quad e\ (1 - e) - (E - e) \geq 0$

This last expression can be rewritten as:

$e - e^2 - E + e = -\ \{\ e^2 - 2e + E\ \} = -\ \{\ (\ e - 1\ )^2 + E - 1\ \} < 0\ \forall\, c_1$

■(Lemma 8.1 - 2)

Lemma 8.1 tells us that if $E(c_0^{min}) \leq 1$ and c_1 is 'small' (or if $E(c_0^{min}) \leq 1$), $\underline{c} = c_0^{min}$ is a candidate to optimum. Nevertheless, to assure it really is so,we need some kind of concavity. Under some conditions we will have it. For $E(c_0^{min}) \leq 1$ and 'big ' c_1 we know $\underline{c} = c_0^{min}$ is not optimal.

Then the optimum (which we know exists) is an interior one. It remains to be seen whether it is unique. Here we will also need to establish concavity conditions.

Up to now we have not ignored any value of $(\bar{x} - \underline{x})$ in the interval (0,1).

It turns out that the concavity conditions are easy to establish when

$(\bar{x} - \underline{x})$ is small enough. In what remains of this section we will work the case of an $(\bar{x} - \underline{x})$ such that $E(c_0^{min}) \leq 1$. In the appendix we will solve the general case.

Fact (8.5) If $\underline{c}$, c_1 are such that $0 < e < 1/2$ and $A \geq 0$ then $\partial FD(\underline{c};c_1)/\partial \underline{c} < 0$

Proof

$\partial FD(\underline{c};c_1)/\partial \underline{c} = \{ 1- e \} V''(\underline{c}) + \{2 e - 1 \}/e^2 - A / e^3 + 2 V'(\underline{c})/e$

The first term is negative because $V''(\underline{c}) \leq 0$ and $0 < e < 1$, the second one because $0 < e < 1/2$, and the third one because $A > 0$ and $e > 0$. Finally, since $V'(\underline{c}) < 0$ and $e > 0$, the fourth term is also negative.

□(Fact 8.5)

Lemma 8.2

Under the assumption $E(\underline{c}_0^{min}) < 1/2$

(i) $c_1 < c_1^* \Rightarrow \underline{c} = c_0^{min}$ *is the unique solution to program* *

(ii) $c_1^* < c_1 \Rightarrow$ *program * has a unique solution, and it is such that*

$\underline{c}^* > c_0^{min}$

Proof

(i) a) $c_1 < c_1^* \Rightarrow FD(c_0^{min}, c_1) < 0$ by lemma 7.1 (i)

Notice now that the objective function is concave and so if we take $\underline{c} > c_0^{min}$ it will remain true that $FD(\underline{c}, c_1) < 0$

Notice also that A and e behave in such a way that they satisfy the conditions fact 8.5 asks to guarantee concavity

b) and c) tell us that in $\underline{c} = c_0^{min}$ such conditions are satisfied

b) $e(c_0^{min}, c_1) > 0$

because $e(c_0^{min}, c_1^{min}) > 0$ and $\partial e / \partial c_1 > 0$

c) $c_1 < c_1^* \Rightarrow A(c_0^{min}, c_1) > 0$ and $e(c_0^{min}, c_1) < E(c_0^{min})$

because c.1) $c_1 < c_1^* \Rightarrow c_1 < c_0^{min} + E(\underline{c}_0^{min})^2/2$

c.2) $A(c_0^{min}; c_0^{min} + E(\underline{c}_0^{min})^2/2) = 0$ and $\partial A / \partial c_1 < 0$

c.3) $e(c_0^{min}; c_0^{min} + E(\underline{c}_0^{min})^2) = E(\underline{c}_0^{min}) < 1/2$

and $\partial e / \partial c_1 > 0$

Now d) tells us that if we increase $\underline{c}$, the conditions remain being satisfied:

d) $\underline{c} < c_1 \Rightarrow A(\underline{c};c_1) > 0$ and $e(\underline{c}; c_1) < 1/2$

follows from the previous point and from

$\partial A / \partial \underline{c} > 0$, and

$\partial e / \partial \underline{c} < 0$

Then $\underline{c} < c_1 < c_1^* \Rightarrow FD(\underline{c}, c_1) < 0$

because $\partial FD(\underline{c}; c_1) / \partial \underline{c} < 0$ since the hypothesis required in fact (8.5) are satisfied.

(We have already seen that $\underline{c} \geq c_1$ is not optimal)

So, $\underline{c}^*(c_1) = c_0^{min}$ is the unique solution if $c_1 < c_1^*$

(ii) From lemma 8.1 we know that if $c_1^* < c_1$, then $\underline{c} = c_0^{min}$ does not solve program *. But we know it does exist a solution. Next we will see it is unique.

a) Consider first c_1 s.t. $c_1^* < c_1 < c_0^{min} + E(\underline{c}_0^{min})^2 / 2$

a.0) $A(c_0^{min}; c_1) > 0$ because $c_1 < c_0^{min} + E(\underline{c}_0^{min})^2 / 2$

and $e(c_0^{min}; c_1) > 0$ since $c_1 > c_1^{min}$

a.1) $FD(c_0^{min}; c_1) > 0$ by lemma 8.1 (ii)

a.2) There exists $\underline{c}^G$, with $c_0^{min} < \underline{c}^G < c^{max}$ s.t. $FD(\underline{c}^G; c_1) < 0$

from fact 8.3

a.3) $\partial FD(\underline{c}; c_1) / \partial \underline{c} < 0$ if $\underline{c} < c_1$

because $\{A(c_0^{min}; c_1) > 0$ and $\partial A / \partial \underline{c} > 0\} \Rightarrow A(\underline{c}; c_1) > 0$

and $\{e(c_0^{min}; c_1) < E(\underline{c}_0^{min}) < 1/2$ and $\partial e / \partial \underline{c} < 0\} \Rightarrow e < 1/2$

because hypothesis required in fact (8.4) are satisfied.

Finally, (a.1), (a.2) and (a.3) imply that the set

$\{ \underline{c} : c_0^{min} < \underline{c} < \min\{ c_0^{max}, c_1 \} \}$ contains a unique $\underline{c}$ s.t. $FD(\underline{c}; c_1) = 0$

b) Consider now c_1 s.t. $c_0^{min} + E(\underline{c}_0^{min})^2 / 2 < c_1 < c_1^{max}$

b.1) $A(c_0^{min}; c_1) < 0$ from lemma 8.1 (iii)

b.2) $\lim_{\underline{c} \to c_0^{max}} A(\underline{c}, c_1) = (\bar{x} - \underline{x}) > 0$

Then, since A is continuous and strictly increasing in $\underline{c}$, there exist a unique $\underline{c}^0$ s. t. $A(\underline{c}^0, c_1) = 0$

b.3) $\underline{c}^0 \leq \underline{c} < c_1 \Rightarrow A(\underline{c}, c_1) > 0 \Rightarrow e(\underline{c}, c_1) < E(\underline{c})$

and thus hypothesis of fact 8.5 are satisfied. So

$\partial\, FD(\cdot) / \partial\, \underline{c} < 0$

Since $FD(\underline{c}^0, c_1) = 0$ and $\lim_{\underline{c} \to c_1} FD(\underline{c}, c_1) = -\infty$

then there is a unique $\underline{c}^* \in [\underline{c}^0, c_1)$ s. t. $FD(\underline{c}^*; c_1) = 0$ and $\underline{c}^*$ is a global maximum.

■(Lemma 8.2)

<u>Analysis of e_1</u>

The behavior of e_1 as c_1 varies is derived from the above solution because

$[\bar{c} - \underline{c}]^2 / 2 = c_1 - \underline{c} \; \forall\, \underline{c}, \bar{c}, c_1$

Thus we have that

$e_1^*(c_1)$ is continuous (differentiable) if and only if $\underline{c}^*(c_1)$ is continuous (differentiable), and

$\text{sign}\; d\, e_1^* / d\, c_1 = \text{sign}\; \{ d\, \underline{c}^* / d\, c_1 + 1 \}$

CASE $c_1^{min} < c_1 < c_1^*$

$d\, e_1^* / d\, c_1 > 0$ because $d\, \underline{c}^* / d\, c_1 = 0$

An increase in the expected utility that has to be guaranteed to the country is obtained solely through an increase in $\bar{c}$, leaving constant $\underline{c}^* = c_0^{min}$, which induces an increase in the first period equilibrium effort.

CASE $c_1^* < c_1$.

In this case $d\, e_1^* / d\, c_1 > 0$ because $d\, \underline{c}^* / d\, c_1 \in (0,1)$:

$\underline{c}^*$ is defined by $FD(\underline{c}^*, c_1) = 0$

By the implicit function theorem we have that

$$d\, \underline{c}^* / d\, c_1 = - \frac{\partial\, FOC(\;) / \partial\, c_1}{\partial\, FOC(\;) / \partial\, \underline{c}}$$

And so we have to prove that

$\partial FD() / \partial c_1 \leq - \partial\, FD(\;) / \partial\, \underline{c}$

i.e., that

$\{ - e^2 [1 + V'(\underline{c})] + e + A \} / e^3 \leq$

$\{ -e^3(1-e)V''(\underline{c}) - e^2 [V'(\underline{c})+1] - e(e-1) - V'(\underline{c})e^2 + A \} / e^3$

which after some algebra turns out to be equivalent to

$(1- e) V''(\underline{c})+ [1 + V'(\underline{c})]/ e \leq 0$

To prove this inequality, we can make use of the two conditions which define an interior optimum:

$e(1-e) [V'(\underline{c}) + 1] = E - e(FD(\underline{c} , c_1) = 0)$

$(1-e) V''(\underline{c}) + 2 [1+ V'(\underline{c})] / e - E / e^3 \leq 0 \quad (\frac{\partial FD}{\partial \underline{c}} (\underline{c} ; c_1) \leq 0)$

It suffices to prove that

$E / e^3 - [1+ V'(\underline{c})] / e \leq 0$

which is equivalent to (substituting $V'(\underline{c})$ and $V''(\underline{c})$ and simplifying)

$2 \underline{c} - (2 \underline{c})^{3/2} / (\bar{x}-\underline{x}) \leq e (1-e)$

This inequality holds because:

i)$e (1-e) > (\bar{x} - M) [1 - (\bar{x} - M)]$

since $(\bar{x} - M) < e < 1/ 2$

to see why it is $(\bar{x} - M) < e$ notice that $e + A = E > 2 (\bar{x} - M)$

and $e > A$ since

$FD (\underline{c} , c_1) = 0$ is equivalent to $A/ e = (1-e) (V'(\underline{c}) + 1)$

$e < 1/ 2$ because $A > 0$ implies $e < E$

ii) $2 \underline{c} - (2 \underline{c})^{3/2} / (\bar{x}-\underline{x}) \in [0, 32 (\bar{x} - M)^2 / 54]$

To see why, define $\underline{c} \equiv b (\bar{x} - \underline{x})^2 / 2$. Remaining in the domain of $\underline{c}$ is equivalent to $b \in [1/4 , 1]$

We have that $2 \underline{c} - (2 \underline{c})^{3/2} / (\bar{x}-\underline{x}) = b(\bar{x} - \underline{x})^2 (1- b^{1/2})$

which achieves its global maximum of $32 (\bar{x} - M)^2 / 54$ when $b = 4/9$

Finally, $(\bar{x} - \underline{x}) \in (0, 1)$ implies $(\bar{x} - M) [1-(\bar{x} - M)] > (\bar{x} - M)^2 / 2$

(This ends the proof that $e'(c_1) > 0$)

■ (Proposition 8)

Proof of proposition 9

CASE $b^B < b^C$

It can be proved, in a similar fashion as in the equal discount factors case, that there is an equilibrium in which the bank's strategy solves the following problem.

$$\max_{\underline{c},\, \bar{c}} \quad \underline{x} + b^B V(\underline{c}) + b^C(\bar{c} - \underline{c})\{\bar{x} - \underline{x} + b^B V(\bar{c}) - b^B V(\underline{c})\}$$

We obtain, by the same reasoning as in the equal discount factors case, that $\underline{c}^* = c^{min}$.

The objective function is concave in $\bar{c}$ and the first order condition is:

$$b^C(\bar{c} - \underline{c})[b^B V'(\bar{c})] + b^C\{\bar{x} - \underline{x} + b^B V(\bar{c}) - b^B V(\underline{c})\} = 0$$

Dividing over $b^C > 0$, we obtain exactly the same condition as in the equal discount factor case, except that instead of having b, we now have b^B.

Thus $\quad (\bar{c}^* - c^{min}) = (\bar{x}-M) / b^B + (\bar{x}-M)^2 / 4$, and

$$e_1^* = b^C(\bar{c}^* - c^{min}) = [b^C / b^B](\bar{x} - M) + b^C(\bar{x}-M)^2 / 4$$

from where the statements about e_1^{2*} follow.

The statements about $e_2^{2*}(\cdot)$ are derived from the facts that $\bar{c}^* > c^{max}$ and $\underline{c}^* = c^{min}$

CASE $b^C < b^B$

We first prove, in the same manner as we did in the equal discount factor case, that we can write the bank's problem as

$$\max_{\underline{c},\bar{c},\bar{R}} \quad \underline{x} + b^B V(\underline{c}) + e_1^*[\bar{R} - \underline{x} + b^B V(\bar{c}) - b^B V(\underline{c})]$$

$$\text{s.a.} \quad e_1^* = \begin{cases} 0 & \text{if } \bar{x} - \bar{R}_1 + b^C(\bar{c} - \underline{c}) \leq 0 \\ \bar{x} - \bar{R}_1 + b^C(\bar{c} - \underline{c}) & \text{if } 0 < \bar{x} - \bar{R}_1 + b^C(\bar{c} - \underline{c}) < 1 \\ 1 & \text{if } 1 \leq \bar{x} - \bar{R}_1 + b^C(\bar{c} - \underline{c}) \end{cases}$$

CASE $e_1^* \in (0,1)$

Fact 9.1 $\quad [\bar{R} - \underline{x} + b^B V(\bar{c}) - b^B V(\underline{c})] > 0$

It follows immediately from the proof of proposition (2) □(Fact 9.1)

<u>Fact 9.2</u> $\underline{c} = c^{min}$ is optimal

(It is proven as in proposition (2)) □(Fact 9.2)

In the proof of proposition 2, at this point we proved $\bar{R} = \bar{x}$ was an optimum. Now this does not happen:

<u>Fact 9.3</u> $c^{min} < \bar{c}^* < c^{max}$

Take an arbitrary e_1^* and fix it. Then, if λ is the multiplier associated to the restriction

$$e_1^* = \bar{x} - \bar{R}_1 + b^C(\bar{c} - \underline{c}),$$

a necessary condition for an optimum is:

$$e_1^* = \lambda$$

$$- e_1^* b^B / b^C V'(\bar{c}^*) = \lambda$$

(Deriving the lagrangean function with respect to $\bar{R}$ and $\bar{c}$, respectively)

from where $\bar{c}^*$ must satisfy $- V' (\bar{c}^*) = b^C / b^B$.

Such $\bar{c}^*$ exists and is unique because

$$\lim_{\bar{c} \longrightarrow c^{max}} V'(\bar{c}) = - 1,$$

$$V'(\bar{c}^{min}) = 0 \quad , \text{ and}$$

$$V''(c) < 0 \text{ si } c \in (c^{min}, c^{max})$$

□ (Fact 9.3)

Finally, deriving with respect to $\bar{R}$ the bank objective function, we obtain:

$$\bar{x} - \bar{R}_1 + b^C (\bar{c} - \underline{c}) = [\bar{R} - \underline{x} + b^B V (\bar{c}) - b^B V (\underline{c})]$$

from where

$$\bar{x} - \bar{R}_1 + b^C (\bar{c} - \underline{c}) = (\bar{x} - M) + \{ b^B [V (\bar{c}) - V (\underline{c})] + b^C [\bar{c} - \underline{c}]\} / 2$$

Now, notice that

$\text{sign} \{ b^B[V(\bar{c}) - V(\underline{c})] + b^C [\bar{c} - \underline{c}] \}$

$= \text{sign} \{ V(\bar{c}) - V(\underline{c}) - V'(\bar{c}^*)(\bar{c} - \underline{c}) \}$

and this last one is positive by the concavity of V (c)

Thus $e_1^{2*} > (\bar{x} - M) = e_1^{1*}$

The statement about $e_2^{2*}(\bar{x})$ follows from $\bar{c}^* \in (c^{min}, c^{max})$

The statement about $e_2^{2*}(\underline{x})$ follows from $\underline{c}^* = c^{min}$

By a similar argument as the one employed in the equal discount factors case, we have that every bank's choice leading to $e_1 = 0$ or to $e_1 = 1$, yields expected profits which are below those in the previously characterized point.

To complete the proof we derive now the first best solution. Maximize the country's expected utility subject to provide the bank with a fix expected utility and a nonnegative consumption to the country in each period. Assume, without loss of generality, that $\underline{x} = 0$.

The problem is

$$\text{Max } e_1(\bar{x} - \bar{R}_1) - (e_1)^2/2 + b^C \{ e_2(\bar{x} - \bar{R}_2) - (e_2)^2/2 \}$$

s.a.

$$e_1(\bar{R}_1 - \underline{x}) + b^B\{ e_2(\bar{R}_2 - \underline{x}) \} = \bar{B}$$

$$\bar{x} - \bar{R}_1 \geq 0$$

$$\bar{x} - \bar{R}_2 \geq 0$$

Then, if λ, μ, ν are the multipliers associated to the first, second and third restrictions, respectively, the following conditions are necessary for an optimum.

$$(\bar{x} - \bar{R}_1) - e_1 = \lambda\,(\bar{R}_1 - \underline{x})$$

$$b^C\{ (\bar{x} - \bar{R}_2) - e_2 \} = \lambda\,(\bar{R}_2 - \underline{x})\, b^B$$

$$- e_1 = \lambda\, e_1 + \mu, \quad \mu \geq 0$$

$$- e_2\, b^C = \lambda\, b^B\, e_2 + \nu, \quad \nu \geq 0$$

which are obtained deriving the lagrangean function with respect to e_1, e_2, $\bar{R}_1$ and $\bar{R}_2$, respectively.

From the above conditions we have:

i) If $b^B < b^C$, the first best effort levels are $e_1 = (\bar{x} - \underline{x})\, b^C/b^B$, and $e_2 = (\bar{x} - \underline{x})$

<u>Proof</u> (of i)

Note first that, since $b^C/b^B > 1$, it must be the case that $\mu > 0$. Otherwise, we would have $\lambda = -1$ and $\lambda = -b^C/b^B - \nu/b^B e_2 < -1$, which is a

contradiction. Then the restriction $(\bar{x} - \bar{R}_1) = 0$ is satisfied with equality. Now, since we consider points of the utility frontier in which the country's expected utility is non-negative, it must be that $(\bar{x} - \bar{R}_2) > 0$. So $\nu = 0$, from where $\lambda = -b^C/b^B$. Substituting λ in the first two necessary conditions we obtain the values for e_1 and e_2

□(i)

ii) If $b^C < b^B$, the first best effort levels are: $e_1 = (\bar{x} - \underline{x})$, $e_2 = (\bar{x} - \underline{x})\, b^B / b^C$

Proof (of ii)

Notice that, since $b^c / b^B < 1$, it must be that $\nu > 0$ since, otherwise, we would have $\lambda = -1 - \mu < -1$ and $\lambda = -b^C / b^B$, which is a contradiction. Then the restriction $(\bar{x} - \bar{R}_2) = 0$ is satisfied with equality. Since we consider points in the utility frontier in which the country's expected utility in non-negative, it must be that $(\bar{x} - \bar{R}_1) > 0$. So $\mu = 0$, from where $\lambda = -1$. Substituting this value of λ in the first two conditions we obtain the effort levels we mentioned above.

□(ii)

■ (Proposition 9)

Proof of proposition 10

CASE $b^B_1 < b^B_2 \leq b^C$

We know that

$$e_1^{2*} = [\, b^C / b^B]\, [\, \bar{x} - M \,] + b^C\, [\, \bar{x} - M \,]^2 / 4$$

$$\equiv [\, b^C / b^B]\, [\, \bar{x} - M \,] + [\, b^C / 2 \,]\, \{\, V(\, \bar{c}\,) + \bar{c} - V(\, c^{min}) - c^{min} \,\}$$

EU country $= c^{min} + (\, e_1^{2*} \,)^2$

EU bank $= \underline{x} + b^B\, V(\, c_0^{min}\,)$

$$+ b^C\, (\, \bar{c}^* - \underline{c}_0^{min}\,)\, [\, \bar{x} - \underline{x} + b^B\, V(\bar{c}^*) - b^B\, V(\, c_0^{min}\,)\,]$$

Thus

$d\, e_1^{2*} / d\, b^B = -[\, b^C / (b^B)^2 \,]\, [\, \bar{x} - M \,] < 0$, and thus (iii) follows.

EU country is increasing in e_1^{2*} and thus d EU pa!s / d $b^B < 0$ and (i) follows.

Finally d EU bank / d $b^B = V(\, c_0^{min}\,)\, (\, 1 - e_1^{2*}) + e_1^{2*} V\, (\, \bar{c}^*\,) > 0$ and thus (ii) follows.

CASE $b^C < b_1^B < b_2^B$

We know that

$$e_1^{2*} = (\bar{x} - M) + \{ b^B [V(\bar{c}^*) - V(c^{min})] + b^C [\bar{c}^* - c^{min}] \} / 2$$

$$\text{EU country} = c_0^{min} + (e_1^{2*})^2$$

$$\text{EU bank} = \underline{x} + b^B V(c_0^{min}) + e_1^{2*}[\bar{R}^* - \underline{x} + b^B V(\bar{c}^*) - b^B V(c_0^{min})]$$

Thus:

$$d e_1^{2*} / d b^B = \{ V(\bar{c}^*) - V(c_0^{min}) + b^B V'(\bar{c}^*) d\bar{c}^* / d b^B + b^C d\bar{c}^* / db^B\} / 2$$
$$= [V(\bar{c}^*) - V(c_0^{min})] / 2 + d\bar{c}^* / d b^B [b^B V'(\bar{c}^*) + b^C] / 2$$

The first term is negative because $V(\cdot)$ is decreasing, the second one is zero by the optimality condition for $\bar{c}^*$.

Thus, $d e_1^{2*} / d b^B < 0$, and so (iii) follows.

Since EU country is increasing in e_1^{2*}, d EU pa!s / d $b^B < 0$, and we have (i)

Finally,

$$d\, \text{EU bank} / d b^B = V(c_0^{min}) [1 - e_1^{2*}] + e_1^{2*} V(\bar{c}^*) > 0,$$ and so (ii) follows.

CASE $b_1^B \leq b^C < b_2^B$

Take an arbitrary b^C and consider $b_1^B \leq b^C < b_2^B$.

We must prove that

$$e_1^{2*}(b_1^B) = (\bar{x} - M)\frac{b^c}{b_1^B} + \frac{b^C}{2}\{ V(\bar{c}_1) + \bar{c}_1 - V(c^{min}) - c^{min} \} >$$
$$(\bar{x}-M) + \frac{b_2^{\ B} [V(\bar{c}_2) - V(c^{min})] + b^C [\bar{c}_2 - c^{min}]}{2} = e_1^{2*}(b_2^B)$$

To do it, notice that $b^C / b_1^B \geq 1$ implies the first term on the left-hand side is greater than the first one on the right hand side.

As for the second term, first notice that

$$b^C[V(\bar{c}_2) - V(c^{min})] > b_2^{\ B}[V(\bar{c}_2) - V(c^{min})]$$

because

$$b^C < b_2^B \text{ y } [V(\bar{c}_2) - V(c^{min})] < 0$$

Thus

$$b_2^{\ B}[V(\bar{c}_2) - V(c^{min})] + b^C [\bar{c}_2 - c^{min}] < b^C [V(\bar{c}_2) - V(c^{min}) + \bar{c}_2 - c^{min}]$$

On the other hand,

$$b^c [V(\bar{c}_2) - V(c^{min}) + \bar{c}_2 - c^{min}] < b^C [V(\bar{c}_1) - V(c^{min}) + \bar{c}_1 - c^{min}]$$

because

$$\partial [\ \bar{c} + V(\ \bar{c}\ )\ ] \,/\, \partial \bar{c} \;=\; 1 + V'(\ \bar{c}\ ) > 0 \text{y} \quad \bar{c}^{*}\ (b_1^B) > c^{max} > \bar{c}^{*}(\ b_2^B)$$

Thus:

$$e\ (\ b_1^B\ ) \quad > \quad e\ (\ b_2^B\ )$$

Likewise:

(i) EU country under b_1^B > EU country under b_2^B, and

(ii) EU bank (b_2^B) > EU bank ($\bar{R} = \bar{x}$, $\bar{c}_1^{*}$, c^{min} ; b_2^B) >

EU bank ($\bar{R} = \bar{x}$, $\bar{c}_1^{*}$, c^{min}, b_1^B)

where the first inequality is due to the fact that { $\bar{R} = \bar{x}$, $\bar{c}_1^{*}$, c^{min} } is not the optimal choice under b_2^B and the second one due to $b_2^B > b_1^B$ and all the payments being the same under all state of nature and their probabilities too

■(Proposition 10)

Proof of proposition 11

Remember that we have defined $TC_{t.r}(\mathbf{TR},\mathbf{Te},Th_{t.r})$ as the country's expected utility evaluated at t.r, when the history $Th_{t.r}$ has been observed and from then on the strategies **TR** ,**Te** are followed. We have

Fact 11.1

Under every OSECT, **TR** ,**Te**, ∀ t.2, the country's expected utility and its optimal effort can be written as a function of the following three variables: $\bar{R}_t$, $TC_{(t+1).1}(\mathbf{TR},\mathbf{Te},Th_{(t+1).1})$ if $x_t = \bar{x}$ occurs, and $TC_{(t+1).1}(\mathbf{TR},\mathbf{Te},Th_{(t+1).1})$ if $x_t = \underline{x}$ occurs. For notational convenience we will call these expected utilities $\bar{c}_{t+1}$ and $\underline{c}_{t+1}$, respectively

Proof

Since under every OSECT the country must behave optimally at each date, at date t.2 the country's expected utility under an OSECT is[1]

$$TC_{t.2}(\ ) = \text{Max} \begin{cases} b\underline{c}_{t+1} + \\ e^T_t \{ (\bar{x} - \bar{R}^t_t) + b\,[\bar{c}_{t+1} - \underline{c}_{t+1}] \} - (e^T_t)^2/2 & \text{if } \underline{x} < \bar{R}^t_t \\ (\underline{x} - \bar{R}^t_t) + \\ b\underline{c}_{t+1} + e^T_t \{ \bar{x} - \underline{x} + b[\bar{c}_{t+1} - \underline{c}_{t+1}] \} - (e^T_t)^2/2 & \text{if } \bar{R}^t_t \le \underline{x} \end{cases}$$

Since the above function is concave in e^T_t, the optimal effort satisfies:

$$e^T_t(\cdot) = \begin{cases} b(\bar{c}_{t+1} - \underline{c}_{t+1}) + (\bar{x} - \bar{R}^t_t) & \text{if } \underline{x} < \bar{R}^t_t \\ \\ \bar{x} - \underline{x} + b[\bar{c}_{t+1} - \underline{c}_{t+1}] & \text{if } \bar{R}^t_t \le \underline{x} \end{cases} \tag{11.1}$$

and substituting in $TC_{t.2}(\cdot)$ we have:

---

[1] Notice that $\bar{c}$ ($\underline{c}$) has been defined as the country's expected utility derived from the pair of strategies that form an OSECT if at t $\bar{x}$ ($\underline{x}$) occurs.

$$TC_{t.2}(\cdot)=\begin{cases} b\,\underline{c}_{t+1}(\,) + \{\bar{x} - \bar{R}^t_t + b\,(\bar{c}_{t+1} - \underline{c}_{t+1})\}^2/2 & \text{if } \underline{x} < \bar{R}^t_t \\ (\underline{x} - \bar{R}^t_t) + b\,\underline{c}_{t+1} + \{\bar{x} - \underline{x}\, b[\bar{c}_{t+1} - \underline{c}_{t+1}]\}^2/2 & \text{if } \bar{R}^t_t \leq \end{cases} \tag{11.2}$$

□(Fact 11.1)

We make the assumption that when faced with a proposal that leaves it with the same utility as the previous bounds, the country accepts.

Fact 11.2.

The country's strategy in every OSECT specifies to accept every proposal such that $TC_{t.2}$ under proposal $\geq$ $TC_{t.2}$ under previous bounds
(Obvious)

□(Fact 11.2)

Lemma 3.1

$\forall$ t $\leq$ T, under any OSECT there exists C_t^{Tmin} such that the country's expected utility evaluated at t.2, over the remaining hoizon to complete T periods C_t^T, is not lower than C_t^{Tmin}. Furthermore, the bank's expected utility at t.1 over the remaining horizon to complete the T periods, can be written as a function $V_t^T(\cdot)$ of C_t^T.

$V_t^T(\ ): [C_t^{Tmin}, \infty) \longrightarrow [\ V_t^{Tmax}, -\infty)$

and satisfies:

a) $V^T(c) = \{\ \underline{x} + (\bar{x} - \underline{x})^2 / 2\ \}\ (1 - b^{T-t+1}) / (1-b) - c$

if $c > \{ (\bar{x}-\underline{x})^2 / 2 \}(1-b^{T-t+1}) / (1-b)$

b) $V_t^T(\cdot)$ is decreasing.

c) $[\ V_t^T(c+h) - V_t^T(c)\ ] / h \leq 1$

d) $\{\ \bar{x} + \underline{x} + b\ [\ c_t^T(0) + V_t^{T\ max} - c_t^{T\ min}]\ \} / 2 < \bar{x} + b\ \underline{x}\ (1 - b^{T-t+1}) / (1-b)$

where V_t^{Tmax} is the bank's expected utility at the beggining of the t-th period when there is no restriction at all over the expected utility the country must obtain

Proof (Lemma 11.1)

Property (a)

We first see that property (a) holds:

$V_t^T(c) \leq \{\ \underline{x} + (\bar{x} - \underline{x})^2 / 2\ \}\ (1 - b^{T-t+1}) / (1-b) - c \quad \forall c$

because $\{\ \underline{x} + (\bar{x} - \underline{x})^2 / 2\ \}\ (1 - b^{T-t+1}) / (1-b)$ is the expected utility the country gets when it totally internalizes the consequences of its decisions. When $c > \{ (\bar{x}-\underline{x})^2 / 2 \}(1-b^{T-t+1}) / (1-b)$, the bank can be repaid $\underline{x}$ each period without causing any distortion and then the previous weak inequality is satisfied with equality.

□(Property a)

To prove the other properties we will proceed by induction on the number of periods remaining to be played.

For $t = T$ we are in the last period of a two period model and thus we can verify that the lemma is satisfied if we define $V^T_T(\ ) \equiv V_2(\ )$, where $V_2(\cdot)$ is the function we used for the last period in the two period model.

Assume that for period $(t + 1)$ (with $2 \leq t + 1 \leq T$) all the statements in the lemma are true. We will show that then in period t they are also true.

First notice that the bank's expected utility over the remaining horizon to complete T periods can be written as a function of only $\{ \bar{R}^t_t, \bar{c}_{t+1}, \underline{c}_{t+1} \}$. (For notational convenience in the rest of this proof we will write $\bar{c}$, $\underline{c}$ instead of $\bar{c}_{t+1}$, $\underline{c}_{t+1}$)

Assume an expected utility of c must be guaranteed to the country in t.2. Then, if the bank's strategy maximizes its expected utility at t.1 it must solve the following

Progam 11

$$V^T_t(c) = \text{Max}\begin{cases} \underline{x} + bV^T_{t+1}(\underline{c}) + \\ \{ b(\bar{c} - \underline{c}) + (\bar{x} - \bar{R}) \}\{ \bar{R} - \underline{x} + b[V^T_{t+1}(\bar{c}) - V^T_{t+1}(\underline{c})] \} & \text{if } \bar{R} > \underline{x} \\ \bar{R} + b\ V^T_{t+1}(\underline{c}\ ) + \\ \{ b(\bar{c} - \underline{c}) + (\bar{x} - \underline{x}) \}\{ b[V^T_{t+1}(\bar{c}) - V^T_{t+1}(\underline{c})] \} & \text{if } \bar{R} \leq \underline{x} \end{cases}$$

$$\text{s.t.} \qquad \bar{c},\ \underline{c} \geq C^{T\ min}_{t+1}$$

$$\bar{R}^t_t \leq \bar{x}$$

$$C^T_{t\ .2}(\bar{c}\ ,\ \underline{c}\ , \bar{R}_t) \geq c \qquad \text{(restriction 11)}$$

where $C^{T\ min}_{t+1}$ is the minimum expected utility for the country under any OSECT at $(t + 1)$.

That is, the bank must make a proposal $\{ \bar{R}^t_t, \bar{c}, \underline{c} \}$ that maximizes its expected utility subject to being accepted by the country and to $\bar{c}$ and $\underline{c}$ being credible.

Notice that the function $V^T_t(c\ )$ is well defined, that is, that there is a maximum for each c:

The objective function is continuous because property (c) of $V^T_{t+1}(\cdot)$ implies that it satisfies a Lipschitz condition and is, therefore, uniformly continuous.

To have a compact set we require that:

$\bar{c}, \underline{c} \leq x_{0},$

where x_0 is a bound high enough so as not to be restrictive. Then we can apply Weierstrass theorem to guarantee that there exists a maximum $\forall\ c \geq C_t^{Tmin}$.

Property (c)

Let us see that $V_t^T(\ c\ )$ possesses property (c).

Assume we want to increase c from c_2 to c_1, with $c_1 - c_2 = B > 0$

We can always apply the following method:

Let $B = \bar{c}_1 - \bar{c}_2 \geq \bar{V}_{t+1,2} - \bar{V}_{t+1,1}$

$B = \underline{c}_1 - \underline{c}_2 \geq \underline{V}_{t+1,2} - \underline{V}_{t+1,1}$

where the inequalities are due to property (c) of $V^T_{t+1}(\cdot)$

We then have:

i) the initial effort $\{\ b(\bar{c} - \underline{c}) + (\bar{x} - \bar{R})\ \}$ is constant

ii) c increases in B

iii) V decreases in less than B

So $V_t^T(\cdot)$ possesses property (c)

□(Property c)

Existence of $\underline{C}_t^{Tmin}$

We know see that there exists a value C_t^{Tmin} such that the country's expected utility, evaluated at t.2, over the remaining horizon to complete T periods, TC_t^T, is greater or equal to C_t^{Tmin}.

To prove the above statement, we will solve program (11) without restriction 11.3.

Step 1.

At date t.1 of the T period model it is optimal to set:

i) $R_t^t = \bar{x}$

ii) $\underline{C}_t^T = C_{t+1}^{T\ min}$

Proof:

i) follows from property (c) as applied to $V^T_{t+1}(\ )$.

(ii) follows from $V^T_{t+1}(\cdot)$ being decreasing implies that the objective function is decreasing in $\underline{C}^{T}_{t+1}$

□(Step 1)

We have transformed the problem into a one in one variable, $TC^{T}_{t+1} \equiv \bar{c}$

Step 2.

If at period (t+1) of the T period model occurs:

$\{ \bar{x} + \underline{x} + b [c^{T}_{t+1}(0) + V^{T\,max}_{t+1} - c^{T\,min}_{t+1}] \} / 2 <$

$\bar{x} + b\ \underline{x}\ (1- b^{T-t})/ (1-b)$

then a solution to the bank's optimization problem at point t.1 of the T period model is to set $\bar{R}^{t}_{t} = \bar{x}$,

$\bar{c}^{T}_{t+1} = c^{T\,min}_{t+1}$ and $\bar{c}_{t+1}$ such that it is the case that

$b\ V^{T}_{t+1}(\bar{c}_{t+1}) + \bar{x} = \{ \bar{x} + \underline{x} + b [c^{T}_{t+1}(0) + V^{T\,max}_{t+1} - c^{T\,min}_{t+1}] \} / 2$

(11.4)

Proof

When maximizing at t.1 we first establish $\bar{R}^{t}_{t} = \bar{x}$, $\underline{c}^{T}_{t+1} = c^{T\,min}_{1}$ (Step 1)

We assume now that we face the function $V^{T}_{t+1}(c) = c^{T}_{t+1}(0) - c$. Under this assumption, (11.4) is the first order condition to the bank's problem, a concave problem in $\bar{c}$. Then, by properties (a) , (c) and (d), we know it also solves the original problem.

□(Step 2)

We have found a solution to the bank's problem when we ignore the restriction 11. It is easy to see that every solution provides the country with the same expected utility: instead of $\bar{R}^{t}_{t} = \bar{x}$, it is possible to set $\bar{R}^{t}_{t} < \bar{x}$ if it is accompanied by $\bar{c}_{t+1}$ in the region in which $V^{T}_{t+1}(\ )$ is defined by

$$V^{T}_{t+1}(c) = \{ \underline{x} + (\bar{x} - \underline{x})^{2} / 2 \} (1- b^{T-t+2}) / (1-b) - c$$

but this change also leaves the country with the same expected utility. Let us call it C^{Tmin}_{t}. Then we have that, under any OSECT, the country's expected utility in period t, over the remaining horizon to complete T periods, is no lower than C^{Tmin}_{t}.

□(Existence of C^{Tmin}_{t})

Property (b)

We now establish that the function $V^{T}_{t}(c)$ evaluated in $c > C^{Tmin}_{t}$ is decreasing.

Assume we start from a value of c which comes from $\{ \bar{R}^t_t, \bar{c}, \underline{c} \}$, and consider the behavior of $V^T_t(\cdot)$ when we want to achieve a small reduction in c. By property (c) of V^T_{t+1} we can assume, without loss in generality, that we start from $\bar{R}^t_t = \bar{x}$.

i) if $\underline{c} > C^{Tmin}_{t+1}$ we can choose

$\underline{c}' = \underline{c} - h$

$\bar{c}' = \bar{c} - h,$

with $0 < h < \underline{c} - C^{Tmin}_{t+1}$, so that

e is constant

$V^T_{t+1}(\underline{c}) \geq V^T_{t+1}(\underline{c})$ and $V^T_{t+1}(\bar{c}') \geq V^T_{t+1}(\bar{c})$

since $V^T_{t+1}(\cdot)$ is nonincreasing

So, $V^T_{t+1}(y) \geq V^T_{t+1}(c)$ for $c - [\underline{c} - C^{Tmin}_2] \leq y < c$

ii) if $\underline{c} = C^{Tmin}_{t+1}$, then the bank's expected utility has a maximum at $\bar{c}^*$ such that $TC_{1.2}(C^{Tmin}_{t+1}, \bar{c}^*, \bar{x}) = C^{Tmin}_t$ and is concave in $\bar{c}$. Then it is decreasing in $\bar{c}$ for $\bar{c} \geq \bar{c}^*$.

Then if we reduce $\bar{c}$ leaving $\underline{c}$ and $\bar{R}^1_1$ constant, we obtain a reduction in c and an increase in the bank's expected utility.

□(Property b)

Property d

It only remains to prove property (d). This property causes that, once we have set $\underline{c} = c^{min}$ and $\bar{R} = \bar{x}$, the optimal value for $\bar{c}$ is the the 'linear region'.

From step 2 we know that a solution to the bank's problem can be interpreted as a one period maturity debt:

$\bar{R}^t_t = \bar{x}$ means that all the first period product is paid to the bank.

Let $D^T_t \equiv \{ \bar{x} + \underline{x} + b [c^T_{t+1}(0) + V^{T\,max}_{t+1} - c^{T\,min}_{t+1}] \} / 2$

then notice that if in the t-th period $\bar{x}$ occurs and is entirely paid to the bank, the above statement says that in the (t+1)-th period the expected utility for the bank will be $(D^T_{t+1} - \bar{x}) / b$ and this utility is lower (this is the hypothesis of the lemma) than the expected value of always receiving $\underline{x}$. That is, this utility can be obtained setting a repayment that is always received with probability one.

Notice now that if we apply step 2 to compute D^T_t and D^T_{t+1} we will obtain:

$$D^T_t - b\ D^T_{t+1} = M\ (1-b) + b\ \{\ c^1_1(0) + \underline{x} + (e^{T-1}_1) / 2\ \}^2 / 2 > \underline{x}$$

Then if it occurs $\underline{x}$, since $(\ D^T_t - \underline{x}) / b > D^T_{t+1}$, the bank must concede an additional forgiveness if we want to interpret the amounts D as one period maturity debts.

Thus, as long as hypothesis (d) is fulfilled, we can write the optimal bank's strategy as the proposition says.

In step 3 we will find the behavior of e^T_1 when hypothesis (d) is fulfilled, and in step 4 we find (sufficient) conditions for this hypothesis to be held $\forall\ T \in N$.

<u>Step 3</u>

If hypothesis (d) is fulfilled, then $\forall\ T \in N$, under every OSECT:

(a) $e^{T+1}_1 = (\bar{x} - M)(1 - b) + b\ (\bar{x} - \underline{x})^2 / 4 + b\ e^T_1 - 3\ b\ (e^T_1)^2 / 4$, y

(b) $e^T_1 < e^{T+1}_1 < 0.5744\ (\bar{x} - \underline{x})$

<u>Proof.</u>

The first order condition of the bank's problem in the (T + 1) period model when hypothesis (d) (when referred to the first period of a T period model) is fulfilled can be expressed:

$$e^{T+1}_1 / b = (\bar{x} - M)/b + [\ c^T_1(0) - V^{T\ max}_1 - c^{T\ min}_1] / 2 \qquad (11.5)$$

$$= (\bar{x}-M)/b + \{\ b\ [c^{T-1}_1(0) - V^{T-1\ max} - c^{T-1\ min}] +$$

$$+ c^1_1(0) - [\ c^{T\ min}_1 + V^{T\ max} - b\ c^{T-1\ min} + b\ V^{T-1\ max}\ ]\ \} / 2$$

because $c^T_1(0) = b\ c^{T-1}_1(0) + c^1_1(0)$

and if hypothesis (d) is fulfilled for the first period of a (T − 1) period model we can, using (11.5) , substitute e^T_1 and write:

$$e^{T+1}_1 / b = (\bar{x}-M)(1-b) / b + e^T_1 + \{\ \underline{x} + (\bar{x}-\underline{x})^2 / 2 - [\underline{x} + 3\ (e^T_1)^2 / 2\ ]\ \} / 2$$

because $c^{T\ min}_1 = b\ c^{T-1\ min}_1 + (e^T_1)^2$, and

$$V^{T\ max}_1 = \underline{x} + b\ V^{T-1\ max}_1 + (e^T_1)^2$$

and thus statement (a) follows.

To prove statement (b), notice that

(i) There exists a unique $\bar{e}$ such that

$$\Phi(e) \equiv (\bar{x}-M)(1-b) + b(\bar{x}-\underline{x})^2/\ 4 + b\ \bar{e} - 3b(\bar{e})^2/\ 4 - \bar{e} = 0 \qquad (11.6)$$

Furthermore, $\bar{e} \in (\bar{x} - M\ ,\ 2\ /\ 3)$

To see why, note that when considering the left-hand side of (11.6) as a function of $\bar{e}$, $\phi(e)$, it is continuous, strictly decreasing, evaluated in $(\bar{x} - M)$ is strictly positive and evaluated in 2/ 3 is strictly negative.

□(i)

(ii) $e_1^T < \bar{e} \quad \Rightarrow \quad d\ e_1^{T+1}/\ d\ e_1^T \ > \ 0$

since by equation (a) we obtain:

$$d\ e_1^{T+1}/\ d\ e_1^T = b\ (1-\ 3\ e_1^T\ /\ 2)$$

and by (i) we know that $e_1^T < \bar{e} \Rightarrow \ e_1^T < 2\ /\ 3$

□(ii)

(iii) $d\ e_1^{T+1}\ /\ d\ e_1^T < 1$ (obvious)

(iv) e_1^T monotonically converges to $\bar{e}$

since by (i), (ii) and (iii) it follows:

$$e_1^T < \bar{e} \ \Rightarrow \ e_1^{T+1} > e_1^T \ , \text{ and}$$
$$e_1^T > \bar{e} \ \Rightarrow \ e_1^{T+1} < e_1^T$$

□(iv)

(v-i) Is is possible to express $\bar{e}$ as a function of b and we have $d\bar{e}\ /\ db > 0$:

In the left-hand side function of equation (11.6), we have $\phi\ '(\bar{e}) > 0$, and thus we can apply the implicit function theorem.

$$\text{Furthermore } \ d\bar{e}/db \ = - \frac{\partial\phi(e,b)\ /\partial b}{\partial\phi(e,b)\ /\partial e} \quad > 0$$

because the numerator is positive and the denominator negative, and so we can put the following bound to $\bar{e}$:

(vi) $\bar{e} \leq \ 0.5744\ (\bar{x} - \underline{x})$

since this is the value it takes when $b = 1$.

□(Step 3)

Step 4.

$$\text{Let } D_1^{T+1} \equiv \{ \bar{x} + \underline{x} + b \, [\, c_1^T (0) + V_1^{T\ \max} - c_1^{T\ \min} \,] \, \} / 2$$

$$\text{and let } m_T \equiv D_1^T - \bar{x} - b \, \underline{x} \, (1-b^{T-1}) / (1-b) \quad \text{para} \quad T \geq 2$$

Assume hypothesis (d) is fulfilled for the first period of the one period through T period models (in this notation $m_T \leq 0$ is such hypothesis for the (T-1) period model)

Thus

$$m_T = b^{T-1}(e_1^1)^{2} + \dots\dots\dots + b^2 (e_1^{T-2})^2 + b \, (e_1^{T-1})^2 + e_1^T - (\bar{x} - \underline{x})$$

Proof

By definition, for T = 2:

$$m_2 \equiv D_1^2 - \bar{x} - b \, \underline{x}$$

If for T = 1 hypothesis (d) is fulfilled, then

$$D_1^2 = \underline{x} + b \, \underline{x} + b \, (e_1^1)^2 + e_1^2$$

from where

$$m_2 = b \, (e_1^1)^2 + e_1^2 - (\bar{x} - \underline{x})$$

Thus, for the two period model the lemma holds.

We will prove now that if it holds for the (T-1) period model, then it also holds for the T period model:

From definition of m_T we have:

$$m_T = D_1^T - \bar{x} - b \, \underline{x} - b^2 \, \underline{x} \, \dots\dots - b^{T-1} \, \underline{x}$$

If we employ such definition for m_{T-1} and substitute we will get

$$m_T = b \, m_{T-1} - (1- b) \, \bar{x} - b \, \underline{x} + (D_1^T - b \, D_1^{T-1})$$

and if in the models with T - 2 and T - 1 periods hypothesis (d) is fulfilled:

$$D_1^T - b \, D_1^{T-1} = M \, (1-b) + b \, \{ \, c_1^1(0) + \underline{x} + e_1^{T-1} / \, 2 \, \}^2 / 2$$

by step (2) applied to both periods, after rearranging terms.

Thus we have

$$m_T = b \, m_{T-1} - (1-b) \, (\bar{x}-M) + b \, \{ \, c_1^1(0) - \underline{x} + (e_1^{T-1})^2 \, \} / 2$$

and since

$c_1^1 (0) = \underline{x} + (\bar{x} - \underline{x})^2 / 2$, we obtain

$$m_T = b \, m_{T-1} - (1-b)(\bar{x} - M) + b \, (\bar{x} - \underline{x})^2 / 4 + b \, (e_1^{T-1})^2 / 4$$

and using step (3)

$$m_T = b \, m_{T-1} + e_1^T - b \, e_1^{T-1} + b \, (e_1^{T-1})^2 - (1-b) \, (\bar{x} - \underline{x})$$

Assume now (induction hypothesis) that this lemma is valid for the (T - 1) period model. Then:

$$m_T = b^{T-1}\,(e_1^1)^2 + \ldots\ldots + b^2(e_1^{T-2})^2 + b\; e_1^{t-1} - b\;(\bar{x} - \underline{x}) + e_1^T - b\; e_1^{T-1}$$
$$+\, b\;(e_1^{T-1})^2 - (1-b)\;(\bar{x} - \underline{x})$$

and then, simplifying, it follows that the lemma is valid for the T period model.

□(Step 4)

Step 4 allows us to find (sufficient) conditions under which hypothesis (d)

is fulfilled $\forall\; T \in N$, but also (sufficient) conditions under which there exists a T from which they do not hold any longer.

a) $m_T \leq \bar{e}\;[1 + b + b^2 + \ldots + b^{T-1}] - (\bar{x} - \underline{x}) \leq$
$$\bar{e}\;/\;(1-b) - (\bar{x} - \underline{x}) \leq$$
$$0.5744\;(\bar{x} - \underline{x})\;/\;(1-b) - (\bar{x} - \underline{x}) = (\bar{x} - \underline{x})(b - 0.4256)\;/\;(1-b)$$

and thus

$$b \leq 0.4256 \quad \Rightarrow \quad m_T \leq 0 \qquad \forall\; T \in N$$

On the other hand:

b) $m_T \geq (e_1^1)^2\;[1-b^T]/(1-b) - (\bar{x} - \underline{x}) \quad \geq \quad \{\;(\bar{x}-M)^2 - 2(\bar{x}-M)(1-b)\;\}/\;(1-b)$

the above expression has the same sign as

$$\{\;(\bar{x} - M) - 2\;(1-b)\;\}\;/\;(1-b)$$

and thus

Si $b \geq 1 - \;(\bar{x} - \underline{x})/\;4 \;\Rightarrow\; m_T \geq 0 \qquad \forall\; T \geq t_0$ for some $t_0 \in N$

□(Property d)

□(Lemma 11.1)

From lemma 11.1 it follows (it is obvious from the way it is proved) proposition 11

■(Proposition 11)

APPENDIX 2. CHARACTERIZATION OF EQUILIBRIUM UNDER LIMITED DEBT

We have seen that a complete characterization of an OSEC with initial debts needs to make use of the value of $(\bar{x} - \underline{x})$. We do this in the following propositions.

Proposition 8-a)

There exists $(\bar{x} - \underline{x})^*$ s.t. if $(\bar{x} - \underline{x}) \leq (\bar{x} - \underline{x})^*$, then for the OSEC with initial bounds $\bar{R}_1^{IN}$, $\bar{R}_2^{IN}(R_1)$:

i) (Continuity of $\bar{R}_2^{1*}(\underline{x})$ and e_1^*)

There exists c_1^*, $c_1^{**} \in (c_1^{min}, c_1^{max})$ s.t. $\bar{R}_2^{1*}(\underline{x})$ and e_1^* are continuous an (except in c_1^*) differentiable functions of c_1, with domain $[c_1^{min}, \infty)$

ii) (Behavior of $\bar{R}_2^{1*}(\underline{x})$)

$\bar{R}_2^{1*}(\underline{x})$ satisfies:

$c_1^{min} \leq c_1 \leq c_1^* \Rightarrow \bar{R}_2^{1*}(\underline{x}) = M$

$c_1^* < c_1 < c_1^{max} \Rightarrow \underline{x} < \bar{R}_2^{1*}(\underline{x}) < M$ and $\bar{R}_2^{1*}(\underline{x})$ is strictly decreasing in c_1

$c_1^{max} \leq c_1 \Rightarrow \bar{R}_2^{1*}(\underline{x}) \leq \underline{x}$

iii)(Behavior of $e_1(c_1)$)

Furthermore, there exist $(\bar{x}- \underline{x})^{PP}$ and $(\bar{x} - \underline{x})^P$, with $(\bar{x}- \underline{x})^{PP} < (\bar{x}- \underline{x})^P$ such that

$e_1(c_1)$ satisfies:

iii.i) $e'(c) > 0$ for $c_1^{min} \leq c < c_1^*$

iii.ii) if $(\bar{x} - \underline{x}) \leq (\bar{x} - \underline{x})^{PP}$, then

$e(c_1)$ is strictly increasing in the interval $c_1 \in [c_1^{min}, c_1^{max}]$

iii.iii) If $(\bar{x} - \underline{x}) < (\bar{x}- \underline{x})^P$, then

$e(c_1) < (\bar{x} - \underline{x})$ $\forall c_1 \in [c_1^{min}, c_1^{max}]$

iii.iv) If $(\bar{x} - \underline{x}) = (\bar{x}- \underline{x})^P$,

then $e(c_1^*) = (\bar{x} - \underline{x})$, $e(c_1) < (\bar{x} - \underline{x})$

$\forall c_1 \in [c_1^{min}, c_1^{max})$, $c_1 \neq c_1^*$

iii.v) If $(\bar{x}-\underline{x})^{p} < (\bar{x}-\underline{x}) \leq (\bar{x}-\underline{x})^{*}$, then

$\exists\ c_1^{e(\bar{x}-\underline{x})} < c_1^{*} < c_1^{se(\bar{x}-\underline{x})}$ s.t.

$e\ (c_1^{e(\bar{x}-\underline{x})}) = e\ (c_1^{se(\bar{x}-\underline{x})}) = (\bar{x}-\underline{x})$

$c_1 \in [c_1^{min}, c_1^{e(\bar{x}-\underline{x})}) \cup (c_1^{se(\bar{x}-\underline{x})}, c_1^{max}) \Rightarrow e\ (c_1) < (\bar{x}-\underline{x})$

$c_1 \in (c_1^{e(\bar{x}-\underline{x})}, c_1^{se(\bar{x}-\underline{x})}) \Rightarrow e(c_1) > (\bar{x}-\underline{x})$

That is, in a neighborhood of c_1^{min}, if we increase c_1, $\bar{R}_2^{1*}(\underline{x})$ is so high as credibility allows it to but, if we keep on increasing c_1, the bank finds it optimal start decreasing $\bar{R}_2^{1*}(\underline{x})$, and it does it always continuously. Then $\bar{R}_2^{1*}(\underline{x})$ is first constant and then strictly decreasing in c_1.

As for the e_1 behavior, we have that:

If $(\bar{x}-\underline{x})$ is 'very small ', as we go deeper into a 'debt overhang ' situation - by decreasing the expected utility to the country-, the country's effort decreases continuously until it reaches the equilibrium level of the infinite debt model.

If $(\bar{x}-\underline{x})$ is 'median ', we cannot guaranty that $e_1(c_1)$ is monotone in c_1; but at least we keep $e_1 < (\bar{x}-\underline{x})$ always. This feature disappears if we consider even bigger values of $(\bar{x}-\underline{x})$. In the following proposition we consider higher $(\bar{x}-\underline{x})$ values. We find a new feature: we loose the continuity of $\bar{R}_2^{1*}(\underline{x})$ and e_1 in c_1. We have the following situation.

Proposition 8-b)

There exists $(\bar{x}-\underline{x})^{**} > (\bar{x}-\underline{x})^{*}$ s.t.

If $(\bar{x}-\underline{x})^{*} \leq (\bar{x}-\underline{x}) < (\bar{x}-\underline{x})^{**}$, then $\exists\ c_1^{ch}$ s.t.

$\bar{R}_2^{1*}(\underline{x})$ and e_1^{*} are continuous and differentiable functions of c_1, with domain $[\ c_1^{min}, c_1^{ch}) \cup (c_1^{ch}, \infty)$

$\bar{R}_2^{1*}(\underline{x})$ satisfies:

$c_1^{min} \leq c_1 < c_1^{ch} \Rightarrow \bar{R}_2^{1*}(\underline{x}) = M$

$c_1^{ch} < c_1 < c_1^{max} \Rightarrow \underline{x} < \bar{R}_2^{1*}(\underline{x}) < M$ y $\bar{R}_2^{1*}(\underline{x})$ is strictly decreasing in c_1

$c_1^{max} \leq c_1 \Rightarrow \bar{R}_2^{1*}(\underline{x}) \leq \underline{x}$

$c_1^{ch} = c_1 \Rightarrow \bar{R}_2^{1*}(\underline{x}) \in \{\ M, \lim_{c_1 \to c_1^{ch(+)}} \bar{R}_2^{1*}(\underline{x})(c_1)\ \}$

with $M > \lim_{c_1 \to c_1^{ch(+)}} \bar{R}_2^{1*}(\underline{x})(c_1)$

There exists $c_1^{e(\bar{x}-\underline{x})}$, $c_1^{se(\bar{x}-\underline{x})}$, with $c_1^{e(\bar{x}-\underline{x})} < c_1^{ch} < c_1^{se(\bar{x}-\underline{x})}$ s.t.

$e_1(c_1^{e(\bar{x}-\underline{x})}) = e_1(c_1^{se(\bar{x}-\underline{x})}) = (\bar{x} - \underline{x})$

$c_1 \in [c_1^{min}, c_1^{e(\bar{x}-\underline{x})}) \cup (c_1^{se(\bar{x}-\underline{x})}, c_1^{max}) \Rightarrow e(c_1) < (\bar{x} - \underline{x})$

$c_1 \in (c_1^{e(\bar{x}-\underline{x})}, c_1^{se(\bar{x}-\underline{x})}) \Rightarrow e(c_1) > (\bar{x} - \underline{x})$

The difference with the previous proposition is that the lack of quasi-concavity of the objective function causes $\bar{R}_2^{1*}(\underline{x})$ and e_1^* to take on two different values in c_1^{ch}. Nevertheless, in the domain $[c_1^{min}, c_1^{ch}) \cup (c_1^{ch}, \infty)$ such variables exhibit a continuous behavior in c_1.

Finally, for $(\bar{x} - \underline{x})$ values close to one, since the specification of the model restricts us to effort values $e \leq 1$, it happens that there exist values of c_1 (to be precise in an interval), s.t. the induced effort equals unity, and both $\bar{R}_2^{1*}(\underline{x})$ and e_1 move in such a way that this restriction holds with equality.

Proposition 8-c)

If $(\bar{x} - \underline{x})^{**} \leq (\bar{x} - \underline{x}) < 1$, then:

There exist c_1^{one} and c_1^{ch}, with $c_1^{one} < c_1^{ch}$ s.t.

$\bar{R}_2^{1*}(\underline{x})$ and e_1 are continuous and (except in c_1^{one}) differentiable functions of c_1, with domain $[c_1^{min}, c_1^{ch}) \cup (c_1^{ch}, \infty)$

$\bar{R}_2^{1*}(\underline{x})$ satisfies:

$c_1^{min} \leq c_1 \leq c_1^{one} \Rightarrow R_2^{-1*}(\underline{x}) = M$

$c_1^{one} < c_1 < c_1^{max} \Rightarrow \underline{x} < R_2^{-1}(\underline{x}) < M$ and $R_2^{-1*}(\underline{x})$ is strictly decreasing in c_1

$c_1^{max} \leq c_1 \Rightarrow \bar{R}_2^{1*}(\underline{x}) \leq \underline{x}$

$c_1^{ch} = c_1 \Rightarrow \bar{R}_2^{1*}(\underline{x}) \in \{ \lim_{c_1 \to c_1^{ch(-)}} \bar{R}_2^{1*}(\underline{x})(c_1), \lim_{c_1 \to c_1^{ch(+)}} \bar{R}_2^{1*}(\underline{x})(c_1) \}$

with $\lim_{c_1 \to c_1^{ch(-)}} \bar{R}_2^{1*}(\underline{x})(c_1) > \lim_{c_1 \to c_1^{ch(+)}} \bar{R}_2^{1*}(\underline{x})(c_1)$

There exist $c_1^{e(\bar{x}-\underline{x})}$, $c_1^{se(\bar{x}-\underline{x})}$, with $c_1^{e(\bar{x}-\underline{x})} < c_1^{one} < c_1^{ch} < c_1^{se(\bar{x}-\underline{x})}$ s.t.

$e\ (c_1^{e(\bar{x}-\underline{x})}) = e\ (c_1^{se(\bar{x}-\underline{x})}) = (\bar{x} - \underline{x})$

$c_1 \in [c_1^{min}, c_1^{e(\bar{x}-\underline{x})}) \cup (c_1^{se(\bar{x}-\underline{x})}, c_1^{max}) \Rightarrow e\ (c_1) < (\bar{x} - \underline{x})$

$c_1 \in (c_1^{e(\bar{x}-\underline{x})}, c_1^{se(\bar{x}-\underline{x})}) \Rightarrow e(c_1) > (\bar{x} - \underline{x})$

$c_1 \in [\ c_1^{one}, c_1^{ch}) \Rightarrow e(c_1) = 1$

$c_1 = c_1^{ch} \Rightarrow$ there exist two different equilibrium e_1 values.

<u>Proof of propositions 8 a - c</u>

All the steps taken in the proof of proposition 8 up to lemma 8.1-2 were made for all values of $(\bar{x} - \underline{x})$ in the interval (0, 1). To establish the concavity of the objective function, though, it was necessary to make the assumption of $(\bar{x} - \underline{x})$ being such that $E(c_0^{min}) < 1/2$.
We now establish analogous conditions for the general case.

Quasi-concavity condition

Notice first that

$FD(\underline{c}\ ,\ c_1) = 0 \iff (E - e) / e = (1-e)\ (V'(\underline{c}) + 1)$

Now,

$\partial FD(\underline{c};c_1)/\partial \underline{c} = \{\ 1-\ e\ \}\ V''(\underline{c}) + 2\ /\ e - E\ /\ e^3 + 2\ V'(\underline{c})\ /\ e$

Evaluating $\partial FD(\underline{c};c_1)/\partial \underline{c}$ in $(\underline{c}\ ,\ c_1)$ such that $FD(\underline{c}\ ,\ c_1) = 0$ we obtain

$\partial FD(\underline{c};c_1)/\partial \underline{c} = (1-e)\ \ V''(\underline{c}) + (2e-1)/\ e^2 - (1-e)[V'(\underline{c}) +1]/\ e^2 +$
$2V'(\underline{c})/e$

which has the same sign as

$e^2(1-e)\ V''(\underline{c}) + 2e - 1 - (1-e)\ [V'(\underline{c}) + 1] + 2\ e\ V'(\underline{c})$

this last expression can be transformed into

$e^2(1-e)\ V''(\underline{c}) + [\ V'(\underline{c}) +1][\ 3e-1] -1$

The only term which takes on positive values for $e \in [0,1]$ is

$[\ V'(\underline{c})\ +1][\ 3e-1]$

Then a necessary condition for $\partial FD(\underline{c};c_1)/\partial \underline{c} \geq 0$ is $V'(\underline{c}) \geq -1/2$, because otherwise $[V'(\underline{c}) + 1][3e - 1] \leq 1 \quad \forall\ e \in [0,1]$.

But $V'(\underline{c}) \geq -1/2 \Rightarrow V''(\underline{c}) \leq -27/8\ (\bar{x}-\underline{x})^2$, as it follows from the definition of these two functions: if we want $V'(\cdot)$ to be not too close to minus one, then we must accept $\underline{c}$ to be close to c_0^{min}, which puts an upper bound to $V''(\underline{c})$

Thus

$e^2(1-e)\ V''(\underline{c}) + [\ V'(\underline{c})\ +1][\ 3e-1]\ -1 \geq 0 \ \Rightarrow$

$e^3 - e^2 + (\bar{x} - \underline{x})^2\ 24\ e/27 - (\bar{x} - \underline{x})^2\ 16/27 \geq 0$ (Quasi-concave)

using $V''(\underline{c}) = -27/8\ (\bar{x}-\underline{x})^2$ and simplifying.

Define e^{sup} s.t.

$(e^{sup})^3 - (e^{sup})^2 + 24\ (\bar{x} - \underline{x})^2\ e^{sup}/27 - 16\ (\bar{x}-\underline{x})^2/27 = 0$

$e^{sup}(\bar{x} - \underline{x})$ provides, for each $(\bar{x} - \underline{x})$, an upper bound, such that

$P(\underline{c}\ ,\ c_1)$ is strictly quasi-concave in the set

$\{\ \underline{c}:\ e(\underline{c},\ c_1) \leq e^{sup}\ (\bar{x} - \underline{x})\ \}$

Since $e(\bar{x} - \underline{x})$ is decreasing , we have that the upper bound found for

$(\bar{x} - \underline{x}) = 1$ is lower than the upper bound for every $(\bar{x} - \underline{x})$, and thus

$\{\ \underline{c} : e(\underline{c},\ c_1) \leq e^{sup}\ (1)\ \} \subset \{\ \underline{c} : e(\underline{c},\ c_1) \leq e^{sup}(\bar{x}-\underline{x})\ \} \quad \forall\ (\bar{x}-\underline{x}) < 1$

The function $P(\underline{c}\ ,\ c_1)$ is not always quasi-concave in $\underline{c}$: there exist values for c_1 such that when $FD(\underline{c}\ ,\ c_1) = 0$, $\partial FD(\underline{c}\ ,\ c_1)/\ \partial\ \underline{c} > 0$, which makes the program * harder to solve. What will help us in solving this problem is the fact that when $P(\underline{c}\ ,\ c_1)$ is not quasi-concave, then its third derivative -with respect to $\underline{c}$- is negative. This property will guarantee that if there exist $\underline{c}$ such that $FD(\underline{c}\ ,\ c_1) = 0$, $\partial FD(\underline{c}\ ,\ c_1) > 0$, then it also exist $\underline{c}'$ (and it will be unique), such that it satisfies $FD(\underline{c}',\ c_1) = 0$ and $\partial FD(\underline{c}',\ c_1) \leq 0$

Condition negativeness of the third derivative

Deriving $\partial FD(\underline{c}\ ,\ c_1) \,/\, \partial\ \underline{c}$ with respect to $\underline{c}$ and simplifying, we get

$$\partial^2 FD(\underline{c}\ ,\ c_1) \,/\, \partial\ \underline{c}^2 = (1-e)\ V'''(\underline{c}) + 3\ V''(\underline{c}) \,/\, e + 3\ [1+V'(\underline{c})] / \ e^3 - 3\ E(\underline{c}) \,/\, e^5$$

Only the first term is positive. Then, a sufficient condition for

$\partial^2 FD(\underline{c}\ ,\ c_1) \,/\, \partial\ \underline{c}^2 \leq 0$ is

$(1-e)\ V'''(\underline{c}) \ \leq - 3\ V''(\underline{c}) \,/\, e$

Substituting the functions V''( ) and V'''( ) we obtain

$e\ (1 - e) \leq\ 2\ \underline{c}$

Since $\underline{c} \geq c_0^{min} = (\bar{x} - \underline{x})^2 / \ 8$, a sufficient condition for this inequality to hold is

$e(1-e) \leq (\bar{x} - \underline{x})^2 / \ 4$ (Third derivative)

For each $(\bar{x} - \underline{x})$, (Third derivative) provides a lower bound for e, namely:

$e^{inf}(\bar{x} - \underline{x}) = \max \{ e \text{ t. q.} e(1-e) = (\bar{x} - \underline{x})^2 / \ 4 \}$,

such that $\partial^2 FD(\underline{c}\ ,\ c_1) \,/\, \partial\ \underline{c}^2 < 0$ in the interval such that

$e(\underline{c},\ c_1) \ \geq e^{inf}(\bar{x} - \underline{x})$

We can relate both conditions in the following way:

<u>Fact</u> <u>2.8</u>

If $e \leq e^{sup}(\bar{x} - \underline{x})$, then $P(\underline{c}\ ,\ c_1)$ is strictly quasi-concave

If $P(\underline{c}\ ,\ c_1)$ is not strictly quasi-concave, then $\partial^2 FD(\underline{c}\ ,\ c_1) \,/\, \partial\ \underline{c}^2 < 0$

<u>Proof</u>

Assume $P(\underline{c}\ ,\ c_1)$ is not quasi-concave

a) Assume that $e \leq 64 \,/\, 69$

Since $P(\underline{c}\ ,\ c_1)$ is not quasi-concave, then

$e^3 - e^2 + (\bar{x} - \underline{x})^2\ 24\ e/\ 27 - (\bar{x} - \underline{x})^2\ 16/\ 27 \geq 0$ (quasi-concave)

from where

$e - e^2 \leq (\bar{x} - \underline{x})^2\ \{\ 24/\ 27 - 16/\ 27\ e\ \}$

On the other hand,

$(\bar{x} - \underline{x})^2 \{\ 24/\ 27 - 16/\ 27e\ \} \leq (\bar{x} - \underline{x})^2 / \ 4$ which is equivalent to

$24/\ 27 - 16/\ 27e \leq 1/4$, equivalent to $e \leq 64/69$

From where

$e - e^2 \leq (\bar{x} - \underline{x})^2 / \ 4$ and, thus $\partial^2 FD(\underline{c}\ ,\ c_1) \,/\, \partial\ \underline{c}^2 < 0$

b) Assume now that $e > 64/69$.

Since e is not quasi-concave, then $(\bar{x} - \underline{x}) \geq 0.37$

Finally, $e > 64/69$ and $(\bar{x} - \underline{x}) \geq 0.37 \Rightarrow e - e^2 \leq (\bar{x} - \underline{x})^2 / 4 \Rightarrow$

$\partial^2 FD(\underline{c}, c_1) / \partial \underline{c}^2 < 0$

□(Fact 2.8)

Now we have sufficient elements to characterize the solution to program *. We will proceed in a different way depending on the value of $(\bar{x} - \underline{x})$.

Let $E(c_0^{min}) < 1$ and $(\bar{x} - \underline{x}) \leq 0.892636$

$c_1 \leq c_1^* \Rightarrow c_0^{min}$ is the unique solution to program *

Proof:

a) For $c_1 = c_1^*$, $\underline{c} = c_0^{min}$ is the unique solution:

$FD(c_0^{min}, c_1) = 0$ by definition of c_1^*

and $\partial FD / \partial \underline{c} \, (c_0^{min}, c_1^*) < 0$ iff $(\bar{x} - \underline{x}) \leq 0.892636$

We now state that there does not exist $c' > c_0^{min}$ t.q. $FD(c', c_1^*) = 0$.

Assume there is. If $FD(\underline{c}, c_1^*) \leq 0 \;\forall\; \underline{c} \neq \underline{c}'$, then $FD(\cdot\, ; c_1^*)$ reaches an interior maximum at $\underline{c}'$ and so $\partial FD(\underline{c}'; c_1) / \partial \underline{c} = 0$ which contradicts strict quasi-concavity.

If $\exists\, \underline{c}''$ s.t. $FD(\underline{c}'', c_1) > 0$, then, by the intermediate value theorem, $\exists$ $\underline{c}''' \in (c_0^{min}, \underline{c}'')$ s.t. $FD(\underline{c}''', c_1) = 0$ and $\partial FD(\underline{c}''', c_1) / \partial \underline{c} \geq 0$ which contradicts strict quasi-concavity.

So, $\underline{c}^*(c_1) = c_0^{min}$ is the unique solution if $c_1 = c_1^*$

b) For $c_1 < c_1^*$, $\underline{c} = c_0^{min}$ is also the unique solution:

Consider an alternative candidate $\underline{c}^C > c_0^{min}$. $\underline{c}^C$ will be preferred to c_0^{min} iff

$$P(\underline{c}^C; c_1) - P(c_0^{min}; c_1) = \int_{c_0^{min}}^{\underline{c}^C} FD(\underline{c}\, ; c_1)\, d\underline{c} \geq 0$$

But

$$\frac{\partial}{\partial c_1} \int_{c_0^{min}}^{\underline{c}^c} FD(\underline{c}\ ;\ c_1)\, d\underline{c} = \int_{c_0^{min}}^{\underline{c}^c} \frac{\partial\ FD}{\partial\ c_1} (\underline{c}\ ;\ c_1)\, d\underline{c} > 0$$

because $\frac{\partial FD}{\partial\ c_1}(\ \underline{c}\ ;\ c_1) > 0$

Thus:

[$P(\underline{c}^c;\ c_1) - P(c_0^{min}\ ;\ c_1)$ decreases as c_1 decreases.

Thus, if c_0^{min} is optimum for c_1^*, it will remain so for $c_1 < c_1^*$.

□

If $E(\ c_0^{min}) < 1$ and $c_1^* < c_1$

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there exists a unique $\underline{c}^*\ (c_1)$ s.t. $FD(\underline{c}^*(c_1)\ ,\ c_1) = 0$. $\underline{c}^*\ (c_1)$ is the unique solution to program *

a) Consider first $c_1^* < c_1 \leq c_1^{one}$. Then $FD(c_0^{min},\ c_1) > 0$. c_0^{min} does not solve program *. But we know it does exist a solution. Now we will show it is unique.

Step 1 If $c_1 < c_0^{max}$, then l!m $FD(\underline{c},\ c_1) = -\infty$ (fact 7.8)

If $c_0^{max} < c_1$, $\lim_{\underline{c}\ c_0^{max}} A > 0$ since $A(c_0^{min}\ ,\ c_1) > 0$ and A is increasing in $\underline{c}$ and $\underline{c} < c_1 \Rightarrow e > 0$, thus hypothesis of fact 7.9 hold and we have:

$$\lim_{\underline{c} \to c_0^{max}} FD(\underline{c}\ ,\ c_1) < 0$$

Intuitively, as we increase $\underline{c}$, if we first reach $\underline{c} = c_1$, then FD approaches $-\infty$ (fact 7.8), and if we first reach $\underline{c} = c_0^{max}$, then FD approaches $L < 0$ (fact 7.9)

Step 2 The set $\{\ \underline{c} : c_0^{min} < \underline{c} < \min\{\ c_0^{max}, c_1\}\ \}$ contains a unique $\underline{c}$ s.t.. $FD\ (\underline{c}\ ,\ c_1) = 0$

The existence of such $\underline{c}$ follows from a.1), a.2) and the intermediate value theorem. To see it is unique, assume to the contrary there exist $\underline{c}' < \underline{c}''$, with

$FD(\underline{c}';c_1) = FD(\underline{c}'';c_1) = 0$

If $\partial FD/\partial\ \underline{c}\ (\underline{c}',\ c_1) < 0$, we immediately obtain a contradiction

If $\partial FD/\partial\ \underline{c}\ (\underline{c}',\ c_1) \geq 0$, then $e(\underline{c}',\ c_1) > e^{sup}(\bar{x} - \underline{x})$, from where

$\underline{c} \leq \underline{c}' \Rightarrow \partial^2 FD\ /\ \partial\ \underline{c}\ < 0$, and

$\underline{c} \leq \underline{c}' \Rightarrow \partial\ FD\ /\ \partial\ \underline{c} > 0$

From the two above implications it follows that

$\underline{c} \leq \underline{c}' \Rightarrow FD(\underline{c}) \leq FD(\underline{c}') = 0$, which is a contradiction.

b) Consider now $c_1 > c_1^{one}$, that is, c_1 such that $e\ (c_0^{min},c_1) > 1$.
Then $\exists\ \underline{c} > c_0^{min}$ s.t. $e(\underline{c},\ c_1) = 1$ (it follows from $e(c_1,\ c_1) = 0$ and the intermediate value theorem).

$e\ (\underline{c},\ c_1) = 1$ and $E(\underline{c}) \leq E(c_0^{min}) < 1 \Rightarrow FD(\underline{c}\ ,\ c\ ) > 0$
and so we can proceed in the same way.

Let[1] $E(c_0^{min}) < 1$ <u>and</u> $(\bar{x} - \underline{x}) > 0.892636$

Then

$\exists\ c_1^{ch}$, with $c_1^{ch} < c_1^*$, s.t. $\exists\ \underline{c}^*(c_1^{ch})$ s.t.

$P(\underline{c}^*(c_1^{ch});\ c_1^{ch}) = P(c_0^{min}\ ;\ c_1^{ch}) > P(\underline{c}\ ,\ c_1^{ch})$

$\forall\ \underline{c} \in (c_0^{min},\ \min\{\ c_1,\ c_0^{max}\ \}\ ]\ ,\ \underline{c} \neq \underline{c}^*$

that is, when $c_1 = c_1^{ch}$, $P(\underline{c}\ ;\ c_1)$ reaches its maximum at two different points: c_0^{min} and $\underline{c}^*$. There are two solutions to program *

$c_1 < c_1^{ch} \Rightarrow \underline{c}^*(c_1) = c_0^{min}$ is the unique solution to program *

$c_1 > c_1^{ch} \Rightarrow$ the unique solution to program * satisfies $\underline{c}^*(c_1) > c_0^{min}$

<u>Proof</u>:

a) If $c_1 = c_1^* \Rightarrow \underline{c}^* > c_0^{min}$ is the unique solution to program * , because

i) $FD(c_0^{min},\ c_1^*) = 0$ by definition of c_1^*

ii) $\partial FD/\ \partial\ \underline{c}\ (c_0^{min},\ c_1^*) > 0$. Then $\underline{c}^* > c_0^{min}$. Furthermore, in a neighborhood

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[1] i.e., $0.892636 < (\bar{x} - \underline{x}) \leq 0.899$

of c_0^{min} it is true that

$\underline{c} > c_0^{min} \Rightarrow$ $FD(\underline{c}\ ,\ c_1^*) > 0$, and we can proceed in the same way as in the case $c_1 > c_1^*$ to prove the uniqueness of program *.

Then we have $P(\underline{c}^*(c_1^*)\ ;\ c_1^*) - P(c_0^{min};\ c_1^*) > 0.$

b) $\exists\ c_1^{inf}$, $c_1^{inf} < c_1^*$, s.t. $\underset{\underline{c}}{\mathrm{Max}}\ FD\ (\underline{c}\ ,\ c_1^{inf}) = 0$

It holds $P(\underline{c}^*(c_1^{inf}\ )\ ;\ c_1^{inf}) - P(c_0^{min};\ c_1^{inf}) < 0$

<u>Proof</u> (of b)

$\underset{\underline{c}}{\mathrm{Max}}\ FD\ (\underline{c}\ ,\ c_1^{min}) < 0$, $\underset{\underline{c}}{\mathrm{Max}}\ FD\ (\underline{c}\ ,\ c_1^*) > 0$

and $\underset{\underline{c}}{\mathrm{Max}}\ FD\ (\underline{c}\ ,\ c_1)$ is continuous in c_1 (it is also differentiable because $FD(\underline{c}\ ;\ c_1)$ has an interior optimum)

Now,

$$P(\underline{c}^*(c_1^{inf}\ ;\ c_1^{inf}) - P(c_0^{min}\ ;c_1^{inf}\ ) = \int_{c_0^{min}}^{\underline{c}^*(c_1^{inf})} FD(\underline{c}\ ;c_1^{inf})\ d\underline{c}\ < 0$$

because $FD(\underline{c}\ ;\ c_1^{one}) < 0$ if $\underline{c} \in [\ c_0^{min}\ ,\ \underline{c}^*(c_1^{inf})).$

From (a) and (b) it follows, by the intermediate value theorem, the existence of c_1^{ch}. Now notice that for $c_1 \in (c_1^{inf},\ c_1^{ch})$ the only solution is c_0^{min} and for $c_1 > c_1^{ch}$ the solution is an interior one.

c) Let $\underline{c}^*(c_1)$ be the highest root of $FD(\underline{c}\ ,\ c_1) = 0$
Then

$$P(\underline{c}^*(c_1\ ;\ c_1) - P(c_0^{min}\ ;c_1) \ = \int_{c_0^{min}}^{\underline{c}^*(c_1)} FD(\underline{c}\ ;c_1)\ d\underline{c}$$

and
$$\frac{\partial}{\partial\ c_1} \int_{c_0^{min}}^{\underline{c}^*(c_1)} FD(\underline{c}\ ;c_1)\ d\underline{c} \ = \int_{c_0^{min}}^{\underline{c}^*(c_1)} \frac{\partial}{\partial\ c_1} FD(\underline{c}\ ;c_1)\ d\underline{c} > 0$$

because $FD(\underline{c}^*(c_1)\ ;\ c_1) = 0$

Notice that $e(c_0^{min};\ c_1^*) < E(c_0^{min}) < 1$ and $c_1^{ch} < c_1^*$, and so the restriction

$\bar{c} - \underline{c} \leq 1$ is not binding for $c_1 \leq c_1^{ch}$

Likewise, if $c_1 > c_1^{ch}$, since the optimum is determined by

$FD(\underline{c}^*, c_1) = (1-e)\ (1+V'(\underline{c})) - (E-e)/\ e = 0$ y $E(\underline{c}) \leq E(c_0^{min}) < 1$ the restriction $\bar{c} - \underline{c} \leq 1$ is not binding either.

Consider now the case[2]

$E(c_0^{min}) > 1$

The situation we face is similar to the case $0.892 \leq 0.899$, but we need to make two variations.

The first one is that $FD(c_0^{min}, c_1) < 0\ \forall\ c_1$, so that we cannot use c_1^* to guarantee the existence of a c_1 s.t. c_0^{min} is not the solution to program *.

The other one is that the restriction $\bar{c}(\underline{c}, c_1) - \underline{c} \leq 1$ can be binding at c_0^{min} for some $c_1 \in (c_1^{min}, c_1^{max})$.

Let $c_1^{one} = c_0^{min} + 1/\ 2$, i.e., c_1^{one} is such that $e(c_0^{min}, c_1^{one}) = 1$

It turns out that $(\bar{x} - \underline{x}) \geq 0.7560 \Rightarrow c_1^{max} < c_1^{one}$.

That is, there exist values for c_1 lower than c_1^{max} such that the choice $\underline{c} = c_0^{min}$ is not feasible.

Remember that $c_0^{one}(c_1)$ is defined by $c_0^{one} + 1/\ 2 = c_1$

That is, $c_0^{one}(c_1)$ satisfies, by definition, $e(c_0^{one}(c_1), c_1) = 1$

Thus for $c_1 > c_1^{one}$ the opportunity set is

$\underline{c} \in [c_0^{one}(c_1), c_0^{max}]$, con $c_0^{min} < c_0^{one}$

We have the following result:

$\exists\ c_1^{ch}$, $c_1^{min} < c_1^{ch} < c_1^{max}$ s.t. $P(c_0^{min} ; c_1^{ch}) = P(\underline{c}^*(c_1^{ch}); c_1^{ch})$

If $c_1^{ch} \leq c_1^{one}$ then the assessment referring to $E(c_0^{min}) < 1$ and $(\bar{x} - \underline{x}) > 0.892636$ holds true:

$P(\underline{c}^*(c_1^{ch}); c_1^{ch}) = P(c_0^{min} ; c_1^{ch}) > P(\underline{c} , c_1^{ch})$

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[2] i.e., $(\bar{x} - \underline{x}) > 0.899$

$\forall\ \underline{c} \in (c_0^{min}, \min\{ c_1, c_0^{max} \}]$, $\underline{c} \neq \underline{c}^*$

that is, when $c_1 = c_1^{ch}$, $P(\underline{c}\ ;\ c_1)$ reaches its maximum at two different points: c_0^{min} and $\underline{c}^*$. There are two solutions to program *

$c_1 < c_1^{ch} \Rightarrow \underline{c}^*(c_1) = c_0^{min}$ is the unique solution to program *

$c_1 > c_1^{ch} \Rightarrow$ the unique solution to program * satisfies $\underline{c}^*(c_1) > c_0^{min}$

If $c_1^{one} < c_1^{ch}$, then $\exists\ c_1^{onech}$, $c_1^{onech} > c_1^{one}$, s.t.

$c_1 \leq c_1^{one} \Rightarrow \underline{c} = c_0^{min}$ is the unique solution to program *

$c_1^{one} < c_1 < c_1^{onech} \Rightarrow \underline{c} = c_0^{one}(c_1)$ is the unique solution to program *

$c_1 = c_1^{onech} \Rightarrow$ there are two solutions, $c_0^{one}(c_1^{onech})$ and $\underline{c}^*(c_1^{onech})$, to program *.

$c_1^{onech} < c_1 \Rightarrow$ the unique solution is $\underline{c} = \underline{c}^*(c_1)$

Step 1. We shall prove the existence of c_1^{ch} in the same way we did it in for the case $0.8926 < (\bar{x} - \underline{x}) < 0.899$

a) $\exists\ c_1^{sup}$, with $c_1^{min} < c_1^{sup} < c_1^{max}$ s.t.:

$P(\underline{c}^*(c_1^{sup})\ ;\ c_1^{sup}) - P(c_0^{min}\ ;\ c_1^{sup}) > 0$

$P(\underline{c}^*(c_1^{sup})\ ;\ c_1^{sup}) - P(c_0^{one}(c_1^{sup})\ ;\ c_1^{sup}) > 0$

To see why, consider $P(c_0^{max}\ ;\ c_1) - P(c_0^{min}\ ;\ c_1)$, and

$P(c_0^{max}\ ;\ c_1) - P(c_0^{one}(c_1);\ c_1)$ as a function of c_1.

It turns out that evaluated at c_1^{max} they are strictly positive because in c_1^{max} the optimal choice is c_0^{max} and $c_0^{one}(c_1) < c_0^{max}$ if $(\bar{x} - \underline{x}) < 1$

Then, by continuity, there is a neighborhood of c_1^{max} in which they are strictly positive. Then, in a neighborhood of c_1^{max}, $\underline{c} \in \{ c_0^{min}, c_0^{one}(c_1) \}$ is not the bank's optimal choice.

b) Now, $P(\underline{c}^*(c_1^{inf})\ ;\ c_1^{inf}) - P(c_0^{min}\ ;\ c_1^{inf}) < 0$, by the same argument applied in the case $(\bar{x} - \underline{x}) > 0.89263$

From a) and b) it follows the existence of c_1^{ch}.

□ (Step 1)

There are two possible cases:

$c_1^{one} < c_1^{ch}$

Then, since by definition $c_0^{min} = c_0^{one}(c_1^{one})$ and, for $c_1 < c_1^{ch}$, the optimal choice is c_0^{min}, we have:

$P(\underline{c}^*(c_1^{one}) \; ; \; c_1^{one}) - P(c_0^{one}(c_1^{one}) \; ; \; c_1^{one}) < 0$, from where

$\exists \; c_1^{chone}$ s.t. $P(\underline{c}^*(c_1^{chone}); \; c_1^{chone}) - P(c_0^{one}(c_1^{chone}) \; ; \; c_1^{chone}) = 0$

Intuitively, if $c_1^{one} < c_1^{ch}$, i.e., if, as we increase c_1, we first arrive to $e(c_0^{min}, c_1) = 1$ than to c_1^{ch} (to c_0^{min} being not optimal) then we will have a zone in which $e = 1$ will be optimal: in which the optimal choice will be $c_0^{one}(c_1)$; but we will come to a point, c_1^{chone}, in which we will leave it.

Now we will prove the uniqueness of c_1^{chone}.

If $c_1^{one} \leq c_1^{ch}$ then for c_1^{one} the unique solution is $c_0^{min} = c_0^{one}(c_1^{one})$.
For $c_1^{one} < c_1$, c_0^{min} is not feasible. The feasible set is $[c_0^{one}(c_1), c_0^{max}]$
We claim that

If $c_0^{one}(c_1)$ is optimal for $c_1 > c_1^{one}$, then $c_0^{one}(c_1')$ is optimal for $c_1^{one} \leq c_1' < c_1$ and c_0^{min} is optimal for $c_1' < c_1^{one}$.

To see why, notice that the difference $P(c_0; c_1) - P(c_0^{one}(c_1) \; ; \; c_1)$ is strictly increasing in c_1 if $c_0^{one}(c_1)$ is optimal:

$$P(\underline{c}_0 \; ; \; c_1) - P(c_0^{one}; \; c_1) = \int_{c_0^{one}}^{\underline{c}_0} FD(\underline{c} \; ; c_1) \, d\underline{c}$$

$$c_0^{one} + 1/2 = c_1 \Rightarrow c_0^{one} = c_1 - 1/2$$

Then

$$\frac{\partial}{\partial c_1} \int_{c_0^{one}}^{\underline{c}_0} FD(\underline{c} \; ; c_1) \, d\underline{c} = \int_{c_0^{one}}^{\underline{c}_0} \frac{\partial}{\partial \underline{c}} FD(\underline{c} \; ; c_1) \, d\underline{c} - \frac{\partial c_0^{one}}{\partial c_1} FD(c_0^{one}, c_1) =$$

$$\int_{c_0^{on}}^{\underline{c}_0} \frac{\partial}{\partial \underline{c}} FD(\underline{c} \; ; c_1) \, d\underline{c} - FD(c_0^{one}; \; c_1)$$

The first term is positive because $e \leq 1$, and the second one because if $c_0^{one}(c_1)$ is optimal, then $FD(c_0^{one} \; ; \; c_1) \leq 0$, So, if $c_0^{one}(c_1)$ is optimal for $c_1 > c_1^{one}$, then $c_0^{one}(c'_1)$ is the unique optimum for $c_1^{one} \leq c'_1 < c_1$. Since, suppose it is not. Then

$P(\underline{c}\ ; c_1) - P(c^{one}; c_1) \leq 0$ and

$P(\underline{c}\ ; c'_1) - P(c^{one}; c'_1) \geq 0$

From where $P(\underline{c}\ ; c) - P(c^{one}; c)$ has an interior minimum in

$c''_1 \in (c'_1, c_1)$, because for values lower than c_1 in a neighborhood c_1,

$P(\underline{c}\ ; c_1) - P(c^{one}; c_1)$ is strictly decreasing.

c''_1 satisfies:

$P(\underline{c}\ ; c''_1) - P(c^{one}; c''_1) < 0$,

(because it is smaller than $P(\underline{c}\ ; c_1) - P(c^{one}; c_1)$

$FD(c_0^{one}(c''_1); c''_1) > 0$ (because c''_1 is a critical point)

but $FD(\cdot\ ;\ ) > 0$ contradicts the optimality of $c_0^{one}(c''_1)$, i.e., contradicts $P(\underline{c}\ ; c''_1) - P(c^{one}; c''_1) < 0$

$c_1^{ch} \leq c_1^{one}$

In this case we obtain the same result as in $(\bar{x} - \underline{x}) > 0.89263$.

It only remains to prove that $\underline{c} \geq c_0^{one}(c_1)$ is never binding, i.e., that we never arrive to the point where $e = 1$. We cannot employ the same proof as in $E(\underline{c}) < 1$.

Assume there exists c'_1, with $c'_1 > c_1^{ch}$, s.t. $P(\underline{c}\ ; c'_1) - P(c^{one}; c'_1)$, i.e., s.t. $c_0^{one}(c_1)$ is optimal.

Then for c_1^{ch}, $c_0^{one}(c_1^{ch})$ is the unique optimum, which contradicts the fact that $c_1^{ch} < c_0^{one}$.

Finally, we claim that if for $(\bar{x} - \underline{x})$ it occurs that $c_1^{one} \leq c_1^{ch}$, then for $(\bar{x} - \underline{x})' > (\bar{x} - \underline{x})$ it also occurs so. To see why, first notice that we can prove, in the same way we proved the optimality of $e = (\bar{x} - \underline{x})$, but using $c_1 = c_0^{min} + [\odot\ (\bar{x} - \underline{x}\ )]^2/\ 2$,
that if for $(\bar{x} - \underline{x})$, $\underline{c} = c_0^{min}$ yields higher expected profits than any other $\underline{c}$,

causing in equilibrium $e(c_0^{min}, c_1) = \odot(\bar{x} - \underline{x})$, then the following implication holds:

$(\bar{x} - \underline{x})' > (\bar{x} - \underline{x})$ and $c_1 = c_0^{min} + [\odot(\bar{x} - \underline{x})']^2/2 \Rightarrow c_0^{min}$ is optimum and the equilibrium effort will be $e(c_0^{min}, c_1) = \odot(\bar{x} - \underline{x})'$

Assume now that for $(\bar{x} - \underline{x})$, when $\odot = (1/(\bar{x} - \underline{x}))$, c_0^{min} (associated to $e = 1$) is optimal.

Then, for $(\bar{x} - \underline{x})' > (\bar{x} - \underline{x})$, c_0^{min} will be optimal for c_1 s.t. $c_1 \leq c_0^{min} + [\odot(\bar{x} - \underline{x})']^2/2$ and c_1 feasible, i.e., for all c_1 that, along with c_0^{min}, yields $e \leq 1$.

<u>Analysis of e_1</u>

We have been discovering the behavior of e_1 when solving the model due to the fact that

$[\bar{c} - \underline{c}]^2/2 = c_1 - \underline{c} \quad \forall\, \underline{c}, \bar{c}, c_1$

So, $e_1^*(c_1)$ is continuous (differentiable) if and only if $\underline{c}^*(c_1)$ is continuous (differentiable), and

sign $d\, e_1^* / d\, c_1 =$ sign $\{ d\, \underline{c}^* / d\, c_1 + 1 \}$

To end the proof of the assessments referring to the cases in which $(\bar{x} - \underline{x})$ is'big ', the following two facts are sufficient

<u>Fact 1.9</u>. If for $(\bar{x} - \underline{x})$ occurs that c_0^{min} is the optimal bank's choice when $c_1 = c_0^{min} + (\bar{x} - \underline{x})^2/2$ (inducing then an equilibrium effort $e_1 = (\bar{x} - \underline{x})$), then for $(\bar{x} - \underline{x})' > (\bar{x} - \underline{x})$ it occurs that c_0^{min} is the optimal bank's choice when $c_1 = c_0^{min} + (\bar{x} - \underline{x})'^{\,2}/2$

<u>Proof</u>:

Fix $c_1 = (\bar{x} - \underline{x})^2/8 + (\bar{x} - \underline{x})^2/2$. Notice we have chosen c_1 in such a way that $e(c_0^{min}; c_1) = (\bar{x} - \underline{x})$. A sufficient condition for the optimality of c_0^{min} is that

$$P(\underline{c}; c_1) - P(c_0^{min}; c_1) = \int_{c_0^{min}}^{\underline{c}} FD(\underline{c}; c_1)\, d\underline{c} < 0$$

for all feasible $\underline{c}$

The previous criterion is not useful to work with because when varying $(\bar{x} - \underline{x})$ it also varies c_0^{min}.

So we will use as our decision variable a, defined as

$\underline{c} = a\,(\bar{x} - \underline{x})^2/\,8$, con $a \in [1,4]$

Then, the lower limit of integration will not change when $(\bar{x} - \underline{x})$ changes. Furthermore, since we use $c_1 = (\bar{x} - \underline{x})^2/\,8 + (\bar{x} - \underline{x})^2/\,2$, bank's profits can be written as a function of only $(\bar{x} - \underline{x})$ and a. We will write them, by an slight abuse of notation, as $P(a,(\bar{x} - \underline{x}))$

The problem is then to know if a = 1 is optimal, that is, if

$P(\underline{a}\ ,(\bar{x} - \underline{x})) - P(1\ ,(\bar{x} - \underline{x})) < 0$

When deriving $\int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(\,a\,;(\bar{x} - \underline{x})\,)\,d\,a$ with respect to $(\,\bar{x} - \underline{x}\,)$

we first find that

$$\frac{\partial^2 P}{\partial a\ \partial(\bar{x}-\underline{x})}(\,a\ ;\ (\bar{x}-\underline{x})\,) = \frac{\partial\ FD\ (\ a;(\bar{x}\ -\underline{x})\ )}{\partial\ (\ \bar{x} - \underline{x}\ )}\ (\ \bar{x} - \underline{x}\ )^2/\ 8$$

$$+\ \frac{\partial\ P}{\partial\ c}(a,\ (\bar{x}-\underline{x})\ )\ \ (\ \bar{x} - \underline{x}\ )\ /\ 4$$

where the first term is negative.

If a = 1 is optimal, then $\int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(\,a\,;(\bar{x} - \underline{x})\,)\,d\,a$ is also negative.

So, $\int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(\,a\,;(\bar{x} - \underline{x})\,)\,d\,a < 0 \Rightarrow \int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(\,a\,;(\bar{x} - \underline{x})\,)\,d\,a$

is strictly decreasing in $(\bar{x} - \underline{x})$.

Thus,

$$(\bar{x}-\underline{x})' > (\bar{x}-\underline{x})\ y\ \int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(a;(\bar{x}-\underline{x})\,)da < 0 \Rightarrow \int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(a;(\bar{x}-\underline{x})')d\ a < 0$$

since otherwise:

in $[(\bar{x}-\underline{x})\ ,\ (\bar{x}-\underline{x})']$ there is an interior minimum, $(\bar{x}-\underline{x})''$, which satisfies $\int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(a;(\bar{x}-\underline{x})'')da > 0$ because the first derivative is zero at $(\bar{x}-\underline{x})''$,

but

$\int_1^{\underline{a}} \frac{\partial P}{\partial \underline{a}}(a;(\bar{x}-\underline{x})'')da < 0$ because a = 1 is optimal. #

□(Fact 1.9)

Corollary. $\exists\ c_1 \in (c_1^{min},\ c_1^{max})$ s.t. $\underline{c}^*(c_1) = c_0^{min}$ and $e^*(c_1) > (\bar{x} - \underline{x})$ iff $(\bar{x} - \underline{x}) > 8\ /\ 9$. To see why, notice that if $(\bar{x} - \underline{x}) = 8/\ 9$, $FD(\underline{c}\ ;\ c_0^{min} + (\bar{x}-\underline{x})^2/\ 2) = 0$, that is, $c_1^* = c_0^{min} + (\bar{x}-\underline{x})^2/\ 2$

Now, is it possible to induce $e_1 = (\bar{x}-\underline{x})$ with $\underline{c} > c_0^{min}$? The following fact tells us that the necessary conditions for this to happen hold iff $(\bar{x} - \underline{x}) >$ 8/ 9

Fact 1.10

i) $e(\underline{c}\ ;\ c_1) = (\bar{x} - \underline{x})$ and $(\bar{x} - \underline{x}) < 8\ /\ 9 \ \Rightarrow\ FD(\underline{c}\ ;\ c_1) > 0$

$\forall\ (\underline{c}\ ,\ c_1) \in [\ c_0^{min},\ c_0^{max}\ ]\ X\ (c_1^{min},\ c_1^{max}\ )$

ii) $(\bar{x} - \underline{x}) \geq 8\ /\ 9 \ \Rightarrow\ \exists$ a unique $c_1 \in (c_1^{min},\ c_1^{max})$ s.t. for some $\underline{c} \in [\ c_0^{min},\ c_0^{max}\ ]$, $FD\ (\underline{c}\ ;\ c_1) = 0$ and $e(\underline{c}\ ;\ c_1) = (\bar{x} - \underline{x})$.

Furthermore, it is satisfied $\frac{\partial\ FD}{\partial\ \underline{c}}(\underline{c}\ ,\ c_1) < 0$.

Proof.

$FD(\underline{c}\ ;\ c_1) = 0$ iff $e(1 - e)\ [\ 1 + V'(\underline{c})\ ] = E - e$. Substitute $e = (\bar{x} - \underline{x})$ in the previous expression and make the following change of variable:

Let $\underline{c} = b^2(\bar{x} - \underline{x})^2/\ 2$, con $b \in [\ 1/\ 2\ ,\ 1\ ]$

Now we get an equation in terms of b and $(\bar{x} - \underline{x})$:

$(\bar{x} - \underline{x})(1- (\bar{x} - \underline{x}))\ (1-b)\ /\ b \ = (\bar{x} - \underline{x})^2\ [\ 1\ /\ 2 - b + b^2/\ 2\ ]$

which after some algebra becomes

$(\bar{x} - \underline{x})\ b^3/\ 2 - (\bar{x} - \underline{x})b^2 + (1 - (\bar{x} - \underline{x})/\ 2)\ b + (\bar{x} - \underline{x}) - 1 = 0$

i) The previous expression, evaluated in b = 1/ 2, is negative (zero, positive) if $(\bar{x} - \underline{x}) < (= , >) 8/9$ and evaluated in $b = 1$ is always zero.

ii) It reaches an interior maximum when $b^{*2} - 4b^{*}/3 + 2/3(\bar{x} - \underline{x}) - 1/3 = 0$

from where $b^{*} = 2/3 - [28/9 - 24/9(\bar{x} - \underline{x})]^{1/2}/2$

Such maximum is positive when $(\bar{x}-\underline{x}) = 1$ and negative when $(\bar{x} - \underline{x}) = 1/2$

iii) The derivative of the maximum (a function of only $(\bar{x} - \underline{x})$) is strictly positive.

Thus, there must exist a value of $(\bar{x}- \underline{x})$ s.t. the maximum is zero (negative, positive) for such value (for lower, higher values). This value is $8/9$, because if $(\bar{x}- \underline{x})= 8/9$, then $b^{*} = 1/2$.

The uniqueness follows from checking that, if $(\bar{x} - \underline{x}) \geq 8/9$, then

$(\bar{x} - \underline{x})\, b^3/2 - (\bar{x} - \underline{x})b^2 + (1 - (\bar{x} - \underline{x})/2)\, b + (\bar{x} - \underline{x}) - 1$

is positive and decreasing in $1/2$, negative and increasing for values close to one, concave for $b \leq 2/3$ and convex for $b > 2/3$

It only remains to prove that when $FD(\underline{c}, c_1) = 0$ and $e(\underline{c}, c_1) = (\bar{x} - \underline{x})$, then $\frac{\partial FD}{\partial \underline{c}}(\underline{c}, c_1) < 0$.

Expressing $\frac{\partial FD}{\partial \underline{c}}(\underline{c}, c_1)$ en terms of b and $(\bar{x} - \underline{x})$ we obtain:

$$\text{sign}\left\{ \frac{\partial FD}{\partial \underline{c}}(\underline{c}, c_1) \right\} =$$

$$\text{sign}\{ (\bar{x}- \underline{x}) - 1 + 2b^2 (\bar{x}- \underline{x}) - 2b^3 (\bar{x}- \underline{x}) - b^3 - 2b^3/(\bar{x}- \underline{x}) + b^4(\bar{x}- \underline{x}) - b^5(\bar{x}- \underline{x})/2 \}$$

Since we are interested in the sign of the previous expression when

$FD(\underline{c}, c_1) = 0$

we can solve for $(\bar{x}-\underline{x})$ in

$$(\bar{x} - \underline{x})\, b^3/2 - (\bar{x} - \underline{x})b^2 + (1 - (\bar{x} - \underline{x})/2)\, b + (\bar{x} - \underline{x}) - 1 = 0$$

and substitute. When we do it we obtain an expression in b which is negative if $b \geq 8/9$: evaluated at $b= 8/9$ is negative and its derivative is negative if $b \geq 8/9$.

□(Fact 1.10)

This is sufficient to prove the convexity of the region in which there is over effort:

Consider increasingly greater values for c_1.

The induced first period effort e_1 equals $(\bar{x} - \underline{x})$ when $c_1 = c_0^{min} + (\bar{x} - \underline{x})^2/2$

For sufficiently small ϵ, if $c_1 = c_0^{min} + (\bar{x} - \underline{x})^2/2 + \epsilon$ then c_0^{min} is still optimal and, thus, $e_1 > (\bar{x} - \underline{x})$

Finally, for c_1 sufficiently close to c_1^{max}, c_0^{min} is not optimal: we have an interior optimum. Furthermore, e_1 approaches (from lower values) $(\bar{x} - \underline{x})$. These facts, along with the uniqueness of the values of c_1 which satisfy the necessary conditions for an interior optimum with $e_1 = (\bar{x} - \underline{x})$, imply that the region that exhibits over effort is an interval.

■ (Propositions 8 a–c)

SERIE DOCUMENTOS DE TRABAJO

The following working papers from recent years are still available upon request from:

Rocío Contreras,
Centro de Documentación, Centro De Estudios Económicos, El Colegio de México A.C., Camino al Ajusco # 20 C.P. 01000 México, D.F.

91/III Young, Leslie and José Romero. "A dynamic dual model of the North American free trade agreement".

91/IV Yúnez-Naude, Antonio. "Hacia un tratado de libre comercio norteamericano; efectos en los sectores agropecuarios y alimenticios de México".

91/V Esquivel, Hernández Gerardo. "Comercio intraindustrial México-Estados Unidos".

91/VI Márquez, Colín Graciela. "Concentración y estrategias de crecimiento industrial".

92/I Twomey, J. Michael. "Macroeconomic effects of trade liberalization in Canada and Mexico".

92/II Twomey, J. Michael. "Multinational corporations in North America: Free trade intersections".

92/III Izaguirre Navarro, Felipe A. "Un estudio empírico sobre solvencia del sector público: El caso de México".

92/IV Gollás, Manuel y Oscar Fernández. "El subempleo sectorial en México".

92/V Calderón Madrid, Angel. "The dynamics of real exchange rate and financial assets of privately financed current account deficits"

92/VI Esquivel Hernández, Gerardo. "Política comercial bajo competencia imperfecta: Ejercicio de simulación para la industria cervecera mexicana".

93/I Fernández, Jorge. "Debt and incentives in a dynamic context".