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On Cournot's theory of oligopoly with perfect complements*

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Abstract

This paper provides a thorough characterization of the properties of Cournot's complementary monopoly model (or oligopoly with perfect complements) in a general setting, including existence, uniqueness and the comparative statics effects of entry. As such, this serves to unify various results from the extant literature that have typically been derived with limited generality. In addition, several studies have suggested that Cournot's complementary monopoly model is the dual problem to the standard Cournot oligopoly model. This result crucially relies on the assumption that the firms have no production costs. This paper shows that if the production costs of the firms are different from zero, the nice duality between these two oligopoly settings breaks down. One implication of this breakdown is that, in contrast to the Cournot model, oligopoly with perfect complements can be a game of strategic complements in a global sense even in the presence of production costs.

JEL codes: C72, D43, L13.

Key words and phrases: oligopoly with perfect complements, price competition, horizontal integration, supermodularity.

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1 Preliminaries

1.1 Introduction

It is well known that Cournot's (1838) pioneering book set the stage for a major paradigm shift in economic theory. In a more direct manner, it initiated the formal study of imperfectly competitive markets and provided an overly precocious foretaste of game theory. While his basic model of quantity competition amongst few firms became a workhorse of applied microeconomics and remains one of the dominant models of partial equilibrium analysis, his other oligopoly model has remained only modestly known even to this day. Cournot's complementary monopoly model refers to a market with the following features. Consumers have a downward-sloping demand for a final product or a system that can only be put together after the purchase of n different components, each of which is sold exclusively by a monopoly supplier. Any subset of components other than the full set has no value in itself for any consumer. The n components constitute thus perfect complements, and none of them possesses any substitutes. The only meaningful demand is thus for the overall system or final product, and the relevant price for consumers is the sum of all the prices paid for all the n components.

The presence of a group of monopolists selling goods that are perfect complements probably explains Cournot's original name for the model. Nonetheless, along the lines of modern game theory, this setting is more aptly referred to as oligopoly with perfect complements, since this explicitly recognizes the strategic interdependence between the "monopolists". (In this paper, we shall use either terminology to refer to this model.)

Two historical settings have motivated the conception of this model. The first, put forth by Cournot (1838) himself is the production of brass, the production of which requires two inputs, copper and zinc at the same time, each supplied by a different monopolist. The demand function here stands for the input demand by the producer for the two inputs (ordered in a one-to-one ratio). Another seminal work in oligopoly with perfect complements was developed independently by Ellet in 1839 (Ellet, 1966). The setting that inspired him dealt with how two different individuals who own consecutive segments of a canal decide their tolls to shippers. We shall discuss a number of different applications of this model throughout the paper.

In an early study, Sonnenschein (1968) showed that Cournot's two theories, the standard Cournot oligopoly and oligopoly with perfect complements, are formally equivalent theories when the costs of production of the firms are zero. Indeed, there is a perfect duality between the revenue functions of the two models, with the decision variables being quantities for the former and prices for the latter. In Cournot oligopoly, price is determined by the the sum of the quantities that the firms produce. In oligopoly with perfect complements, the quantity produced is determined by the sum of the prices of the perfect complements. Then, if the demand function and the inverse demand function are the same, quantities in Cournot's oligopoly model lead to precisely the same prices in oligopoly with perfect complements model and vice versa. Thus the two revenue functions write in exactly the same way in terms of the strategies of the players, which is the sense in which Sonnenschein (1968) meant that the models are mathematically equivalent. However, as we shall demonstrate, if one includes the cost structure of the firms in the model, the said equivalence breaks down in general.

There is a fairly extensive literature in industrial organization that deals with various facets of Cournot's complementary monopoly theory. This simple model has been applied to a variety of settings and has been used in multiple policy debates in various areas, including corruption in government services (Shleifer and Vishny, 1993), patents and innovation policy (e.g., Shapiro, 2001), merger theory (Gaudet and Salant, 1992), competition policy (e.g., Gilbert and Katz, 2001), among others. In recent years, renewed and sustained attention to this topic has surfaced in the law and economics literature (e.g. Heller, 1998 and Heller and Eisemberg, 1998) as well as in the public choice literature dealing in particular with property rights (e.g., Buchanan and Yoon, 2000).

With remarkably few exceptions, these different studies share two common features, First, not surprisingly, they typically restrict attention to the usual convenient functional form of linear demand and costs, and thus work with closed-form solutions. Second, they put in evidence the main result concerning Cournot's second theory, namely that integrating the n different monopoly suppliers into a single decision-making entity would actually improve market performance in a win-win manner (for all concerned, including consumers), despite the fact that the resulting entity would then be what one may refer to as a super-monopoly.[1]

[1] Interestingly, this fundamental insight was already described quite clearly by Cournot in his pioneering work.

The main objectives of this paper may accordingly be described as follows. The first is to provide a fairly extensive characterization of the general properties of oligopoly with perfect complements, including basic theoretical preliminaries such as existence and uniqueness of equilibrium points.[2] In part, this amounts to a generalization of the many related results that have appeared in separate contexts over a long period of time. In addition, since the present paper is based on the methodology of supermodular games, this exercise also serves to provide a unifying framework for studies on this model.[3] The second objective is to qualify some conclusions about this model that have drawn close parallels between Cournot's two theories. The duality that Sonnenschein (1968) observed is actually valid only in the absence of production costs. Similarly, based on a linear specification with nice closed-form solutions, Buchanan and Yoon (2000) also draw close analogies of both a qualitative and a quantitative sort between the standard Cournot oligopoly (reflecting the commons in a way to be made precise later) and Cournot's complementary monopoly model (reflecting the anticommons). Here again, it turns out that these rather striking analogies lack robustness in an essential way: By incorporating linear costs of production into the two models, we show that the parallels largely vanish.

In the overall presentation of the results of this paper, we discuss the relationship between oligopoly with perfect complements and Cournot oligopoly. In particular, we point out the divergences that are engendered by the incorporation of the cost structures into the two models. One example of these differences is that while strategic complementarity of output levels in the standard Cournot oligopoly is not possible in the presence of non-trivial costs (Amir, 1996a), the prices charged by the different monopolies in Cournot's complementary monopoly model may well constitute strategic complements to one another, albeit under general but restrictive assumptions on demand and costs.

The rest of the paper is organized as follows. Section 2 provides a precise definition of Cournot's

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Cournot (1838) found that prices are lower and industry profits higher when a multi-product monopoly produces all the goods instead of having n firms producing the goods.

[2]Surprisingly, relative to the standard Cournot model where such studies have a long history (e.g., Novshek, 1985), no general rigorous theoretic analysis of Cournot's second model is available in the microeconomics literature (to the best of our knowledge).

[3]In particular, we invoke basic results and insights that have appeared in the application of these lattice-theoretic tools in industrial organization theory (see Vives, 1990, 1995; Amir, 1996a; and Amir and Lambson, 2000).

second model and the basic existence proofs for both the asymmetric and the symmetric versions of the model. Section 3 conducts a comparative statics analysis of market performance as the number n of components of the system varies. Section 4 provides a generalization of the usual argument in the literature about welfare and profit-enhancing integration. Finally, Section 5 deals with the inclusion of production costs into the Buchanan and Yoon (2000) setting.

1.2 Some economic applications

This subsection discusses some of the applications of the model at hand in various areas of microeconomics. We provide only a short summary here, and refer the reader to the studies themselves for further details and discussion.

One common application deals with patents (see Shapiro, 2001 and Lerner and Tirole, 2004). This is clearly an application of oligopoly with perfect complements if we think of a firm or consumer that wants to develop a new product but might infringe on a number of different patents owned by different parties. Then, the developer has to pay for the usage of all of the patents involved. The patents in this scenario are perfect complements and the consumer needs to buy a license for each one of them.

In their study of corruption in government services, Shleifer and Vishny (1993) discuss the common situation where a private developer needs different permits (e.g., from the fire, water and police departments) to open a new business. This scenario fits into the present model if the officials are assumed to be fully corrupt bribe maximizers.

Feinberg and Kamien (2001) analyze the hold-up problem that can arise in an oligopoly with perfect complements when the acquisition of the multiple parts is sequential. For example, if the government wishes to buy land from different owners in order to build a public project, one owner can wait until the other owners have set prices for their land in order to get a higher benefit for her part of land given that it is necessary for the project. Clearly, the small pieces of land owned by different agents are perfect complements here.

Ellet (1966) uses the metaphor of two different owners of two sequential segments of a road where there are no alternative routes or exits. Gardner, Gaston and Masson (2002) bring this analogy to the real world and apply the oligopoly with perfect complements model to analyze how the Rhine river was tolled in 1254[4].

A classical application of oligopoly with perfect complements is the anticommons problem. This one arises when multiple agents have the right to exclude people from consuming the good (anticommon good) that they own. This is modeled as each owner choosing the price for the anticommon good that maximizes her profit, with the consumers having to pay each of the owners their price in order to use the anticommon good. Buchanan and Yoon (2000) use as an example to illustrate this problem a vacant lot that can be used as a parking lot that has a lower capacity than its open demand. We will return to this example in the last section of the paper.

As shown by the previous examples, oligopoly with perfect complements is a model that has been widely invoked in the literature. Nonetheless, there is a gap in terms of a general characterization of its equilibria, which this paper hopes to fill.

2 The Model and some basic results

This section lays out the basic model of Cournot's complementary monopoly and provides some basic existence and uniqueness results. The general asymmetric case and the symmetric case are considered separately. The reason for this is that, when insisting on minimal structural assumptions on the model, the existence arguments are quite different across the two cases. In addition, beyond the issues of existence and uniqueness, it is convenient to restrict attention to the symmetric case. This is a key simplifying assumption of the analysis, which is further discussed later on.

Recall that an important auxiliary purpose of this paper is to address and partly correct a widespread but imprecise perception in the literature that the two models that were put forward by Cournot himself in 1938 are duals of one another in some fundamental ways. As the basic results relating to this model are derived, we shall provide a brief comparison with the corresponding results for standard Cournot oligopoly, and assess the similarities and the differences. To avoid confusion between the two models, we shall for the most part refer to the present model as oligopoly with perfect complements (instead of the historical and most commonly used name of Cournot's complementary monopoly). Therefore, we reserve the name of "standard Cournot oligopoly model"

[4]During the period 800-1800, 79 different locations served as toll stations along the Rhine river. The rights to collect tolls were granted by the Emperor, who decided the number, location and amount charged at the toll stations.

for Cournot's much more widely used model of quantity competition.

2.1 The Asymmetric Case

Consider an n-firm oligopoly with perfect complements, i.e., a market situation where each of n producers sells one different good as a monopolist, and these goods are totally useless unless purchased together in a fixed ratio to form a final product. W.l.o.g., we assume that this ratio is one-to-one, since we can always appropriately re-normalize the quantities. In other words, consumers wish to purchase a single system with demand function $D(\cdot)$, composed of n different components, each of which is produced and sold by a separate firm acting as a monopoly supplier for that component. This situation is modeled by letting each of the n firms set the price of its own good/component and each consumer buy one unit of each of the n goods, paying the sum of all the prices set by the firms.

This oligopoly with perfect complements is described by (n, K, D, C_i), where n is the number of firms (or goods), K is the maximum price than can be charged for any of the goods in the complementary market[5], $D : [0, \infty) \to [0, \infty)$ is the demand function and $C_i : [0, \infty) \to [0, \infty)$ is firm i's cost function.

Denote the price that the firm under consideration charges by x and the sum of the prices of the remaining $(n - 1)$ firms by y. Let $z = x + y$ represent the total price that a consumer has to pay in order to obtain the system (of all the complementary goods).

Firm i chooses the price $x \in [0, K]$ that maximizes its profit given by

$$\pi_i(x, y) = xD(x + y) - C_i[D(x + y)]. \tag{1}$$

Its reaction correspondence is

$$r_i(y) = \arg\max\{xD(x + y) - C_i[D(x + y)] : 0 \leq x \leq K\}. \tag{2}$$

Alternatively, we can think of the same firm as choosing $z \in [y, y + K]$ given y, in this case, it maximizes its profit given by

$$\tilde{\pi}_i(z, y) = (z - y)D(z) - C_i[D(z)]. \tag{3}$$

[5] The magnitude of K does not play any role in the proofs, so this is assumption is just for convenience.

Define

$$z_i^*(y) = \arg\max\{(z-y)D(z) - C_i[D(z)] : y \leq z \leq y+K\}. \tag{4}$$

Let $\Delta_i(z,y)$ denote the cross-partial derivative of $\tilde{\pi}_i$ with respect to z and y, then

$$\Delta_i(z,y) = -D'(z), \tag{5}$$

which turns out to be the same for all firms, so we can suppress the index i in equation 5.

Throughout the paper, we maintain the following standard assumptions.[6]

(A1) $D(\cdot)$ is continuously differentiable and $D'(\cdot) < 0$, and

(A2) $C_i(\cdot)$ is twice continuously differentiable and $C_i'(\cdot) \geq 0$.

Under assumption (A1), $\Delta(z,y) > 0$ on the lattice

$$\varphi \hat{=} \{(z,y) : 0 \leq y \leq (n-1)K, y \leq z \leq y+K\}.$$

All the proofs are collected in Section 6. The following elementary but key result follows directly from the fact that the profit function $\tilde{\pi}_i$ in (3) satisfies (a strong notion) of increasing differences on the lattice φ, since $\Delta > 0$ (under smoothness assumptions).

Lemma 1 *Assume that the standard assumptions (A1) and (A2) hold. Then, for each n and i, every selection of $r_i(\cdot)$ satisfies the slope condition $\frac{r_i(y')-r_i(y)}{y'-y} > -1$ for all $y' \neq y$.*

Thus, when a firm's rivals all together raise their total price by some amount, the firm may respond by raising or lowering its own price, but in the latter case never by so much that total price ends up going down (relative to the starting point). This property will play a central role throughout the paper.

The central question under consideration in this section is the characterization of respective sufficient conditions on primitives that turn this oligopoly model into a game of strategic substitutes or strategic complements. As will become clear shortly, this issue naturally subsumes the key issue

[6]Due to the use of supermodularity techniques, the smoothness properties of the demand and cost functions are not necessary for most of the results of this paper. Nevertheless, smoothness is assumed for convenience and ease of interpretation.

of existence of a pure-strategy Nash equilibrium (henceforth, PSNE) for this model. This is also true in case the game is submodular since it clearly has the aggregation property (defined by the fact that each payoff depends only on own action and on the sum of all other players' actions).[7]

As the model at hand may be viewed as a special case of Bertrand competition with differentiated products, one would expect the prototypical case to satisfy the strategic substitutes property since the goods are complements in demand.[8] The first result indicates that this expectation is essentially correct in that it is fulfilled under quite a broad scope in terms of the restrictions needed on demand and costs, as captured by the following assumptions on demand and costs.

Theorem 2 *Assume that the standard assumptions (A1) and (A2) hold. Then, for each $n \in N$, if $D(\cdot)$ is log-concave and $C_i(\cdot)$ is convex for all i, the oligopoly with perfect complements is a game of strategic substitutes and there exists a unique PSNE.*

The conditions of Theorem 2 are general enough as to capture most reasonable specifications of Cournot's complementary monopoly in applied settings, including the widely used case of linear demand and costs (see below for such an example). It follows that one can consider this case to represent the prototypical situation for this model.

Despite the fact that the game at hand is a special case of Bertrand competition with complementary products, it turns out that the scope for strategic complementarity of this game is clearly non-trivial, as evidenced by the following (non-degenerate) sufficient conditions.

Theorem 3 *Assume that the standard assumptions (A1) and (A2) hold. Then, for each $n \in N$, if $D(\cdot)$ is log-convex and $C_i(\cdot)$ is concave for all i, the oligopoly with perfect complements is a game of strategic complements and there exists a (not necessarily unique) PSNE.*

[7] A pure-strategy Nash equilibrium for this model might be referred to in a number of different ways, either as a Cournot equilibrium since the concept goes all the way back to the early book by Cournot (1838) or as a Bertrand equilibrium since it deals with a form of price competition. Nonetheless, to avoid a potential for confusion, we shall retain the neutral name of PSNE.

[8] In Singh and Vives (1984), where the case of Bertrand competition with linear demand for differentiated products is considered in some detail, the properties of strategic substitutes and strategic complements coincide exactly with the properties of the goods being complements or substitutes in demand, respectively. However, for non-linear demands, this is no longer true.

Log-convexity of demand is a rather restrictive condition. Of the commonly used examples, only hyperbolic demand $D(z) = 1/z^\alpha$ with $\alpha > 0$ is log-convex. The limit case of a log-convex demand function is the exponential demand, given by $D(z) = e^{-z}, z \geq 0$, which is strictly convex, log-linear, thus (weakly) log-concave and log-convex.

Observing that the assumptions in the previous results are all in their weak form (as opposed to their strict form), it follows as a direct corollary of the two Theorems that if demand is exponential (i.e., $D(z) = e^{-z}$, $z \geq 0$) and the cost function is linear, then the resulting game must be of both strategic substitutes and of strategic complements. In other words, the reaction curves of all players must be constant functions. We report this formal Corollary in the form of an example.

Example 1. Consider an oligopoly with perfect complements with n firms/goods and a demand function $D(z) = e^{-z}$, $z \geq 0$. Suppose that firm i faces a linear cost function $C_i(q) = c_i q \geq 0$ for producing any output $q \geq 0$. It is easy to derive the reaction curve of firm i (when rivals' total price is $y \geq 0$) as

$$r_i(y) = c_i + 1 \text{ for any } y \geq 0.$$

In other words, each firm has a dominant strategy to price with a mark up of 1 (independent of the actions of the firm's rivals), thus leading to a unique PSNE price vector $(c_1 + 1, c_2 + 1, ..., c_n + 1)$. Consumers pay the total price of $n + \sum_{i=1}^n c_i$ and each firm has equilibrium profit equal to $e^{-(n+\sum_{i=1}^n c_i)}$. As mentioned, this example serves as an illustration of Theorems 2-3 as well as Lemma 1.

With the general conditions for the existence of PSNE in hand, this ends our consideration of the general case. Henceforth, we shall consider the symmetric case (with identical firms). In particular, this will allow us to conduct comparative statics on the effects of exogenously changing the number of firms based on lattice programming methods, with the number of firms being the relevant parameter.

Comparing these existence results to those for standard Cournot oligopoly, many similarities exist, but also one major difference. The latter model can enjoy strategic complementarities in a global sense only in the absence of (non-trivial) costs of production. In other words, while a similar duality as the one reflected in the above results holds for the revenue function of Cournot firms, it does not quite extend to the entire profit function. One consequence of this is that, when facing an exponential inverse demand, Cournot firms have dominant strategies if and only if there are no variable costs in production. For more details on these points, see Amir (1996a).

2.1.1 A special case

In this section, we consider the special case with linear costs, that is, $C_i(D) = c_i D$ with $c_i \geq 0$ for all $i = 1, ..., n$. Firm i's profit becomes

$$\pi_i(x, y) = (x - c_i)D(x + y).$$

The price-cost margin (mark-up) of firm i is given by $m_i = x - c_i$. A change of variable in the previous equation gives us

$$\hat{\hat{\pi}}_i(m_i, m_{-i}) = m_i D\left[m_i + c_i + m_{-i} + c_{-i}\right],$$

where $m_{-i} = \sum_{j \neq i} m_j$ and $c_{-i} = \sum_{j \neq i} c_j$, a symmetric game in price-cost margins.

Let $s = m_i + m_{-i}$ denote the total mark-up, then

$$\hat{\pi}_i(s, m_{-i}) = (s - m_{-i})D(s + c_i + c_{-i}),$$

and $\frac{\partial^2 \hat{\pi}_i(s, m_{-i})}{\partial s \partial m_{-i}} = -D' > 0$. That is, $\hat{\pi}_i$ is supermodular in (s, m_{-i}). By Topkis' Theorem, $s^*(m_{-i})$ is increasing in m_{-i}, which means that m_i^* has slopes greater or equal that -1 in m_{-i}. As a consequence, $m_i^*(m_{-i})$ intersects the line $\frac{m_{-i}}{n-1}$, and we have a symmetric PSNE in price-cost margins, which gives us an asymmetric PSNE back in prices. In other words, all the firms charge the same mark-up in this asymmetric game but different prices, as noted by Cournot (1838).

2.2 The Symmetric Case

Since each firm faces one and the same demand function for its (firm-specific) good or component, to make all firms identical entails only the standard requirement for a symmetric oligopoly that the firms have the same cost function for the production of their respective goods, denoted then by $C : [0, \infty) \to [0, \infty)$.[9] However, since these goods are not homogeneous in any way, the meaning

[9] For ease of notation, whenever we refer to any of the variables or equations defining this model for now on, we will drop the (firm) index i.

of identical cost functions is quite different from the standard one (say for Cournot oligopoly). It typically does not entail access to the same technology, but rather that the different goods or components cost the same to produce for the same number of units (given the postulated one-to-one composition ratio).

The next Theorem establishes that the standard assumptions alone are sufficient to guarantee the existence of at least one symmetric PSNE for the symmetric oligopoly with perfect complements, and no asymmetric PSNEs.

Theorem 4 *Assume that the standard assumptions (A1) and (A2) hold. Then, for each $n \in N$, the oligopoly with perfect complements has at least one symmetric equilibrium and no asymmetric equilibria.*

When all firms are identical, the property captured in Lemma 1, that a firm's reaction curve has all of its slopes bounded below by -1, is alone sufficient to yield existence of a (necessarily symmetric) PSNE. This has no counterpart in the asymmetric version of the model.

Recall that a similar property holds in symmetric Cournot oligopoly in terms of output adjustment following a change in rivals' total output, but not universally so. Indeed, in Amir and Lambson (2000), the corresponding property holds only when production enjoys either decreasing returns to scale or a limited form of scale economies.

3 On the effects of varying the number of components

A proper study of this oligopoly model requires a good understanding of the effects that added or reduced competition would have on equilibrium prices, per-firm output and profit. Although we are asking how changes in the number of firms n affect these equilibrium variables, as in Amir and Lambson (2000), the meaning of the question is somewhat different here. Instead of simple entry or exit by one firm, the issue here is a comparison between the two situations where the exact same final product that consumers want can be produced with either n or $(n + 1)$ components, with each firm's cost function for a component being the same in both cases. Depending on the precise context, the actual economic interpretation of this exercise may in fact reflect quite different scenarios. For instance, in the context of a group of patents, it could be that one of the component patents expires (thus implying a move from n to $n-1$ patents) or that a new patent is added to the group (a move from n to $n+1$ patents).[10]

While this specific question (involving intermediate values of n) has not really been addressed in the literature on Cournot's complementary monopoly, the comparison between monopoly and the n-firm oligopoly is frequently assessed in specific formulations of this model (we shall have more on this below). The answers provided here correspond to what one would expect, on the basis of the specific formulations analyzed so far, in particular one with linear demand and costs.

As uniqueness of PSNE need not prevail for this model, we denote the equilibrium set for each variable by its corresponding capital letter indexed by n. So with n firms, the equilibrium sets are X_n for per-firm price, Y_n for the firm's $(n-1)$ rivals' cumulative price, Z_n for total price, Q_n for per-firm output, and Π_n for per-firm output.

We say that an equilibrium set for a specific variable in the model is increasing or decreasing in n, when the maximal and minimal points of the set are increasing or decreasing in n, respectively.[11] These are represented by an upper and a lower bar on the relevant variable, respectively.

Theorem 5 *Under standard assumptions (A1) and (A2), for each $n \in N$,*

(a) The equilibrium total price Z_n is increasing in n; hence equilibrium per-firm output Q_n is decreasing in n.

(b) The equilibrium per-firm profit Π_n is decreasing in n.

In oligopoly with perfect complements, the addition of one component to the system (or final product) that perfectly complements the existing ones always increases the equilibrium total price. This is very intuitive since now, there is an additional good that the consumer has to buy in order

[10]In the corrupt officials story of Shleifer and Vishny (1993), this might correspond to the government requiring one extra permit (from a new official, say for hygiene) in addition to the existing list of permits. In tolling the Rhine river, it could be that (for a variety of reasons) one of the owning entities decides to offer passage through its own segment toll-free (this then corresponds to a decrease of n by one). Finally, it may be that a policy maker can choose between two technology standards, one involving n components and the other $(n+1)$ components (with each component produced by a monopolist with the same cost function).

[11]This is a well-known feature of comparative statics conclusions based on supermodularity methods (see e.g., Milgrom and Roberts, 1990, 1994, and Echenique, 2002).

to enjoy all of them. Also not surprisingly, the equilibrium profits of each of the existing firms decreases (though the new monopolist increases from no profit to the same profit as all the others).

The results in Theorem 5 have been pointed out before in many particular economic applications, often using particular functional forms. For instance, using the ubiquitous linear demand and zero costs (see Section 5 below), Gardner, Gaston and Masson (2002) show that if the number of segments in road tolling increases, the total price of the tolls goes up while the use of the road and the individual profits of the tolls fall. Moreover, they find that the aggregate profits of the tolls also fall. (The latter result is proved in full generality later on in this paper by Theorem 7.)

Shleifer and Vishny (1993) assert that when there is completely free entry of corrupt officials asking for a bribe to provide a service or good produced by the government, the total bribe approaches infinity, driving the provision of the good and the bribe revenues towards zero.

We now investigate the direction of change of the equilibrium per-firm price X_n. As expected, it can take either direction of change depending on the slope of the reaction curve with respect to rivals' cumulative price. Theorem 6 gives sufficient conditions for these directions of change. Notice that, as can be seen from the proofs, it is an immediate consequence of Theorems 2-4 and Lemma 10 from the Appendix, which states that the equilibrium cumulative price of the rest of the $(n-1)$ firms set is increasing in n.

Theorem 6 *Assume that the standard assumptions (A1) and (A2) hold. Then, for each* $n \in N$,
(a) If $D(\cdot)$ *is log-convex and* $C(\cdot)$ *is concave, the equilibrium per-firm price* X_n *is increasing in* n.
(b) If $D(\cdot)$ *is log-concave and* $C(\cdot)$ *is convex, the (unique) equilibrium per-firm price* x_n *is decreasing in* n.

As reported earlier, the prototypical case for oligopoly with perfect complements is characterized by strategic substitutes, so that per-firm price will have a more pronounced tendency in general to decrease with the number of components.

Now we turn to study what happens to the equilibrium consumer surplus, total profit and social welfare sets when there is an exogenous change in the number of components. By Theorem 5 part (a), the equilibrium total price increases with the number of firms and the equilibrium quantity goes down. Thus, the equilibrium consumer surplus set decreases with more firms in the market.

Recall that by Theorem 5 part (b), the equilibrium individual profit decreases with the number of products or firms. The following result tells us the stronger result that the equilibrium total profit goes down as well. Combining these results, we conclude that equilibrium social welfare is decreasing in n.

Theorem 7 *Assume that the standard assumptions (A1) and (A2) hold, then, for each $n \in N$,*
(a) Equilibrium consumer surplus CS_n is decreasing in n.
(b) Equilibrium total profit $n\Pi_n$ is decreasing in n.
(c) Equilibrium social welfare W_n is decreasing in n.

In the literature on intellectual property rights, many experts have raised the concern that innovation will have a tendency to be stifled in many high-tech industrial sectors by the increasing number of patents. For biomedical research, see Heller and Eisemberg (1998) for more on this. In fact, various calls for a major overhaul of the patenting system are being made both in the U.S. and in Europe.

In some real world examples that fit the setting of oligopoly with perfect complements, the effects captured in this section can lead to dramatically negative consequences for commerce. Shleifer and Vishny (1993) report that, in Zaire, widespread corruption increases the costs of transportation by land so much (due to the large amount of bribes that have to be paid along the way) that it is cheaper to bring the same goods from Europe by ship. In a different but related matter, the excessive number of tolls along the Seine in France around 1400 made shipping costs often more expensive than the goods being transported themselves. In contrast, England was toll free, which is often advanced as a key reason it became the center of commerce (Heilbroner, 1962).

Finally, we extend this analysis to the equilibrium price-cost margin, m_n, defined by $m_n \triangleq x_n - C'[D(z_n)]$. In the empirical literature on market power, this is most often taken as a measure of the level of competition in an industry.

For the model at hand, it turns out that it may increase or decrease with the number of firms depending on whether the demand function is log-convex or log-concave.

Theorem 8 *Suppose that the standard assumptions (A1) and (A2) hold. Then, for any $n \in N$, the equilibrium price-cost margin m_n is decreasing in n if $D(\cdot)$ is log-concave but increasing in n if*

$D(\cdot)$ *is log-convex.*

From a comparison between the results in this Section and those in Amir and Lambson (2000), it is clear that Cournot oligopoly and oligopoly with perfect complements are not mathematically equivalent theories when the firms are symmetric and the production is costly.

Section 5 provides an explicit illustration of this fact.

4 Multi-product monopoly as the integrated solution

In the literature, the main focus is on the comparison between n-firm oligopoly with perfect complements and the corresponding integrated solution wherein one multi-product monopolist offers the entire system (of the same n components) at one overall price. Using the same notation as before, the objective function of this n-product monopolist, who faces an n-fold cost of producing the same amount of each component, is

$$\Pi(x) = xD(x) - nC[D(x)] \tag{6}$$

It is important to observe that the concept of n-product monopolist is different from special case $n = 1$ in the situation considered in the previous section, i.e., where the entire system amounts to a single component. In other words, the n-product monopolist of this section is not obtained when $n = 1$ is substituted in the model of the previous section (indeed, the objective of the latter would then be $\max_x\{xD(x) - C[D(x)]\}$ instead of (6).

The following result compares the market performances of the n-firm oligopoly with perfect complements and of the multi-product monopolist (or the integrated solution).

Proposition 9 *Relative to the n-firm oligopoly with perfect complements, the multi-product monopolist solution leads to*

(a) higher total profits,

(b) a lower total price (and thus higher consumer surplus), and

(c) higher social welfare.

This result is fully intuitive and has repeatedly been reported in different settings, using specific functional forms. With linear demand, see e.g., Buchanan and Yoon (2000) and Gardner, Gaston and Masson (2002). Similarly, Spengler (1950) discusses the gains in efficiency of vertical integration, by avoiding the double-marginalization problem.

5 Multi-product monopoly versus Oligopoly

This section considers the simple framework of Buchanan and Yoon (2000) where Cournot's original two models are compared in a variety of ways under linear demand and costless production. The main purpose here it to establish that the findings in Buchanan and Yoon (2000), which these authors invoked to claim a striking symmetry between the commons and the anticommons, do not carry over to the case of costly production.[12]

5.1 A summary of Buchanan and Yoon (2000)

Buchanan and Yoon (2000) (hereafter, BY, 2000) consider a vacant lot that can be used as a (capacity-constrained) parking. If the vacant lot is a common good, it will be used more than efficiently but if it is privatized, the new $n > 1$ owners will sell permits that the potential users have to buy in order to park in the lot. Any person who wants to park in the vacant lot has to buy one permit from each one of the owners. The outcome is that the vacant lot will be used less than efficiently. The first case illustrates the commons problem and the second one, the anticommons one.

The commons problem can be seen as a Cournot oligopoly with $n > 1$ firms because the owners of the common good decide how much of it to use in order to maximize their profits. Given that the owners cannot exclude others from the usage of the common good, they maximize their profits given the choice of usage of the other owners. The efficient level of usage of the good is equal to the output that a monopolist would choose in this setting; thus, in this case, the relevant concept of monopoly is given by the standard single-product monopolist.

On the other hand, the anticommons problem fits the setting of an oligopoly with perfect complements with $n > 1$ firms. Firms are equivalent to "excluders" that choose the price of their

[12]Costless production is a reasonable assumption in the context of the scenario analyzed by Buchanan and Yoon (2000), as well as in some of the other commonly used situations that are captured by Cournot's complementary monopoly model. However, the typical situation will naturally feature production costs.

permits sold to the potential users in order for them to use the anticommon good. In this case, the relevant concept of monopoly that provides the efficient level of permits (and thus, usage) is a multi-product monopolist that sells all the permits as a bundle (with perfect complements).

BY(2000) solve the equilibrium for the four cases of interest under a linear demand and *zero costs*. With inverse demand $P(q) = a - bq$, a single-product monopoly solves $\max_q q(a - bq)$ and each firm in Cournot oligopoly solves $\max_q q(a - b(q + q'))$ where q' is rivals' total output.

With direct demand $D(z) = \frac{a-z}{b}$, the multi-product monopolist solves $\max_x x(\frac{a-x}{b})$ and a firm in the oligopoly with perfect complements solves $\max_x (\frac{a-(x+y)}{b})$ (with y defined as before).

The first three columns of Table 1 summarize the results in BY(2000).

	BY(2000) (c=0)			Present paper (c>0)		
$n > 1$	Q^*_{BY}	P^*_{BY}	Π^*_{BY}	Q^*	P^*	Π^*
Single-product monopoly	$\frac{a}{2b}$	$\frac{a}{2}$	$\frac{a^2}{4b}$	$\frac{a-c}{2b}$	$\frac{a+c}{2}$	$\frac{(a-c)^2}{4b}$
Cournot oligopoly, n firms	$\frac{na}{b(n+1)}$	$\frac{a}{n+1}$	$\frac{na^2}{b(n+1)^2}$	$\frac{n(a-c)}{b(n+1)}$	$\frac{a+nc}{n+1}$	$\frac{n(a-c)^2}{b(n+1)^2}$
Loss in profit from 1 to n firms			$\frac{a^2(n-1)^2}{4b(n+1)^2}$			$\frac{(a-c)^2(n-1)^2}{4b(n+1)^2}$
Multi-product monopoly, n goods	$\frac{a}{2b}$	$\frac{a}{2}$	$\frac{a^2}{4b}$	$\frac{a-nc}{2b}$	$\frac{a+nc}{2}$	$\frac{(a-nc)^2}{4b}$
Oligopoly with perfect complements, n firms	$\frac{a}{b(n+1)}$	$\frac{na}{n+1}$	$\frac{na^2}{b(n+1)^2}$	$\frac{a-nc}{b(n+1)}$	$\frac{n(a+c)}{n+1}$	$\frac{n(a-nc)^2}{b(n+1)^2}$
Loss in profit from 1 to n firms			$\frac{a^2(n-1)^2}{4b(n+1)^2}$			$\frac{(a-nc)^2(n-1)^2}{4b(n+1)^2}$

Table 1: equilibrium total output (Q^*), equilibrium total price (P^*) and equilibrium industry profit (Π^*) for different settings. The subindex BY stands for the results in BY(2000).

From Table 1, we observe the main results in BY(2000) listed below.

(1) The single product monopolist and the multi-product monopolist produce the same amount of output ($a/[2b]$) and thus, charge the same price ($a/2$) and earn the same profit ($a^2/[4b]$).

(2) Cournot oligopoly produces more than the single-product monopolist ($na/[b(n+1)] > a/[2b]$) and thus, its price is lower ($a/[n+1] < a/2$).

(3) Oligopoly with perfect complements produces less than the multi-product monopoly ($a/[b(n+1)] < a/[2b]$), hence, it charges a higher price ($na/[n+1] > a/2$).

(4) Cournot oligopoly and oligopoly with perfect complements earn the same profit ($na^2/[b(n+1)^2]$).

(5) Each of the two monopolists earn more profit than each oligopolist ($a^2/[4b] > na^2/[b(n+1)^2]$).

(6) The losses in industry profit from changing from single-product monopoly to Cournot oligopoly and from multi-product monopoly to oligopoly with perfect complements are the same ($a^2(n-1)^2/[4b(n+1)^2]$).

Based on the similarity shown in items (1) and (4)-(6), BY(2000) conclude that Cournot oligopoly and oligopoly with perfect complements lead to symmetric tragedies. In particular, these tragedies are reflected in the profit losses of changing from one to n firms having the same magnitude in both models.

5.2 Adding production costs to the BY model

In line with the results of the present paper, we now show that the symmetry in equilibrium industry profits easily breaks down with the inclusion of costs of production. We consider linear costs of production given by $C(q) = cq$ with $c > 0$.

Because the multi-product monopolist produces n goods that are different, it needs n separated plants to produce them and since our setting is symmetric, it pays n times the cost of producing the optimal amount of bundles. Then, the optimization problem of the multi-product monopoly becomes choosing the price x of the bundle that maximizes its profit given by $(x - nc)(\frac{a-x}{b})$. The single-product monopolist, a firm competing à la Cournot and a firm in the oligopoly with perfect complements chooses x that maximizes its profit given by $x(a - bx - c)$, $x(a - b(x + y') - c)$ and $(x - c)(\frac{a-(x+y)}{b})$, respectively, where y' and y are defined as earlier.

The solutions and equilibria to the maximization problems and games listed above are summarized in the last three columns of Table 1 (with $a - nc > 0$). With $c = 0$, we recover the results of

BY(2000). When $c > 0$, we have

(7) The single-product monopoly earns more profits than the Cournot oligopoly ($(a-c)^2/[4b] > n(a-c)^2/[b(n+1)^2]$).

(8) The multi-product monopoly earns more profits than the oligopoly with perfect complements ($(a-nc)^2/[4b] > n(a-nc)^2/[b(n+1)^2]$).

(9) The single-product monopoly earns more profits that the multi-product monopoly ($(a-c)^2/[4b] > (a-nc)^2/[4b]$).

(10) Cournot oligopoly earns more industry profits than the oligopoly with perfect complements ($n(a-c)^2/[b(n+1)^2] > n(a-nc)^2/[b(n+1)^2]$).

(11) The loss in industry profit of changing from single-product monopoly to Cournot oligopoly is bigger than the loss in industry profit of having an oligopoly with perfect complements instead of a multi-product monopoly ($(a-c)^2(n-1)^2/[4b(n+1)^2] > (a-nc)^2(n-1)^2/[4b(n+1)^2]$).

Thus, we have illustrated that the effects on industry profits of adding $(n-1)$ firms to the single-product monopolist are in general, different from the effects on industry profits when we have an oligopoly with perfect complements instead of a multi-product monopoly.

The idea that the tragedies of the commons and anticommons are not symmetric is discussed by Vanneste, Van Hiel, Parisi and Depoorter (2006). Using two experiments, a lab experiment versus a scenario experiment, they conclude that the behaviors of the players facing the same versions of a commons dilemma and an anticommons dilemma are different. In particular, they find that the players act more aggressively (with higher decision variables) when they face the anticommons dilemma. Although this might be explained through the specification of the demand function, this paper brings up the concern that the commons and anticommons problems are not symmetric in general.

6 Proofs

We begin by defining a key mapping for every $n \in N$, which can be thought of as a normalized cumulative best-response correspondence. This mapping is analogous to the one used by Amir and

Lambson (2000) and is useful in dealing with symmetric equilibria in the present context too.

$$B_n : [0, (n-1)K] \longrightarrow 2^{[0,(n-1)K]}$$

where

$$B_n(y) = \frac{n-1}{n}(x' + y). \tag{7}$$

Here, x' represents the firm's best-response, i.e., the price that maximizes its profit in (1) given the cumulative price y for the remaining $(n-1)$ firms. If $x' \in [0, K]$ and $y \in [0, (n-1)K]$, then the (set-valued) range of B_n is as given. Also, a fixed point of B_n, $\hat{y}$, clearly yields a symmetric PSNE where $\hat{x}' = \hat{y}/(n-1)$, i.e., each of the responding firms will set the same price as the other $(n-1)$ firms.

Proof of Lemma 1.

Under (A1), the cross partial derivative of the maximand in (3), $\Delta(z, y)$, is strictly positive on the lattice

$$\varphi = \{(z, y) : 0 \leq y \leq (n-1)K, y \leq z \leq y + K\}.$$

The feasible set $[y, y + K]$ is ascending in y. Then, by a strengthening of the basic monotonicity theorem of Topkis (1978) due to Amir (1996c) and Edlin and Shannon (1998), every selection of z_i^* is strictly increasing in y as long as it is interior.

Since $z_i^*(y) = r_i(y) + y$, it follows directly that $r_i(\cdot)$ has the given slope property.□

Proof of Theorem 2.

By a dual argument to Theorem 3 (proved below), the profit $\pi_i(x, y)$ exhibits the dual single-crossing property under the hypotheses of the Theorem. Since the game at hand is an aggregative game of strategic substitutes, by Tarski (1955) and Novshek (1985), an equilibrium exists.

To show that there is a unique PSNE, use Lemma 1 and the first part of this proof to conclude that all the slopes (of all the selections) of $r_i(\cdot)$ lie in $(-1, 0]$. Then, by a well known (contraction-like) argument, the equilibrium is unique (see e.g., Amir, 1996b). □

Proof of Theorem 3. Milgrom and Shannon (1994) prove, for a Bertrand duopoly with differentiated and complementary products, that if each demand function $\widetilde{D}_i(x, y)$ is log-supermodular and the cost function is concave, then $\pi_i(x, y)$ satisfies the single-crossing property in (x, y). Since log-supermodularity of $\widetilde{D}_i(x, y)$ translates into the log-convexity of $D(x+y)$ in our setting, the proof of this Theorem follows as a special case of their result. Hence, the game is a game of strategic complements and by Tarski's fixed-point Theorem (Tarski, 1955), an equilibrium exists. □

Proof of Theorem 4.
By the proof of Lemma 1, every selection of z^* is increasing in y. Recall that x' denotes the firm's best-response to y, thus $z^*(y) = x' + y$. This implies that for every $n \in N$, every selection of B_n as defined by equation (7) is increasing in y. Then, by Tarski's fixed-point Theorem, (any selection of) B_n has a fixed-point that implies the existence of a symmetric equilibrium of the oligopoly with perfect complements.

Next, we prove that no asymmetric equilibrium can exist. By the proof of Lemma 1, every selection of z^* is strictly increasing. This means that for each $z' \in z^*$ corresponds at most one y such that $z' = x' + y$ (z' is the best-response to y); then, for each total equilibrium price z', each firm must charge the same price $x' = z' - y$, with $y = (n-1)x'$, i.e., no asymmetric equilibrium exists. □

Before proceeding with the rest of the proofs, we need the following intermediate Lemmas.

Lemma 10 *Assume that the standard assumptions (A1) and (A2) hold. Then, for every number of firms $n \in N$, the equilibrium cumulative prices of $(n-1)$ firms set, Y_n, is increasing in n.*

Proof of Lemma 10.
By Topkis's Theorem, the maximal and minimal selections of B_n, denoted by $\overline{B}_n$ and $\underline{B}_n$ respectively, exist. Furthermore, the largest equilibrium cumulative price for $(n-1)$ firms, $\overline{y}_n$, is the largest fixed-point of $\overline{B}_n$. $\overline{B}_n(y)$ is increasing in n for every fixed y. Hence, by Theorem A.4 in Amir and Lambson (2000), the largest fixed-point of $\overline{B}_n$, $\overline{y}_n$, is also increasing in n. Using an analogous argument with $\underline{B}_n$ shows that the the smallest equilibrium cumulative price for $(n-1)$

firms, $\underline{y}_n$, is increasing in n. $\square$

Lemma 11 *Assume that the standard assumptions (A1) and (A2) hold. Then, for every number of firms $n \in N$, $\overline{\pi}_n = \pi(\underline{x}_n, (n-1)\underline{x}_n) \geq \underline{\pi}_n = \pi(\overline{x}_n, (n-1)\overline{x}_n)$.*

Proof of Lemma 11.

We prove that $\overline{\pi}_n = \pi(\underline{x}_n, (n-1)\underline{x}_n)$, a similar argument shows that $\underline{\pi}_n = \pi(\overline{x}_n, (n-1)\overline{x}_n)$. To this aim, observe that $\tilde{\pi}(z, y)$ is decreasing in y, then, $\overline{\overline{\pi}}_n = \tilde{\pi}(\underline{z}_n, \frac{(n-1)}{n}\underline{z}_n) = \pi(\underline{x}_n, (n-1)\underline{x}_n)$. Now, we show that $\overline{\pi}_n = \pi(\underline{x}_n, (n-1)\underline{x}_n)$. Suppose not, then it exists $\tilde{x}_n \in X_n$ such that $\pi(\tilde{x}_n, (n-1)\tilde{x}_n) > \pi(\underline{x}_n, (n-1)\underline{x}_n)$, then, $\pi(\tilde{x}_n, (n-1)\tilde{x}_n) = \tilde{\pi}(\tilde{z}_n, \frac{(n-1)}{n}\tilde{z}_n) > \pi(\underline{x}_n, (n-1)\underline{x}_n) = \overline{\overline{\pi}}_n$, where $\tilde{z}_n = n\tilde{x}_n$, which contradicts the fact that $\overline{\overline{\pi}}_n$ is the maximal element in the set $\tilde{\Pi}_n$, thus, $\pi(\underline{x}_n, (n-1)\underline{x}_n)$ is the maximal per-firm profit equilibrium. $\square$

Proof of Theorem 5.

(a) From the proof of Lemma 1, we know that every selection of z^* is increasing in y. Since $\overline{y}_n$ is increasing in n (Lemma 10), we conclude that $\overline{z}_n$ is increasing in n. Using an analogous argument, $\underline{z}_n$ is increasing in n.

(b) This follows as a direct corollary of the proof of *Theorem 7(b)* where the stronger result $n\overline{\pi}_n \geq (n+1)\overline{\pi}_{n+1}$ is proved. $\square$

Proof of Theorem 6.

(a) By the proof of Theorem 3, the extremal selections from $r(\cdot)$ are increasing in y. Then, $\overline{x}_n = \bar{r}(\overline{y}_n)$, and given that $\overline{y}_n$ is increasing in n (by Lemma 10), so is $\overline{x}_n$. A similar argument follows for $\underline{x}_n$.

(b) By Theorem 2, a unique equilibrium exists which is symmetric by Theorem 4. Also, by the proof of Theorem 2 we know that every selection of $r(\cdot)$ is decreasing in y. Since in equilibrium y_n is increasing in n (by Lemma 10) and $x_n = r(y_n)$, x_n is decreasing in n. $\square$

Proof of Theorem 7.

(a) The consumer surplus, $CS(\cdot)$, at any total price z and for any n is given by

$$CS(z) = \int_z^\infty D(t)\ dt,$$

which is decreasing in z.

Then,

$\overline{CS}_n - \overline{CS}_{n+1} = \int_{\underline{z}_n}^\infty D(t)\ dt - \int_{\underline{z}_{n+1}}^\infty D(t)\ dt = \int_{\underline{z}_n}^{\underline{z}_{n+1}} D(t)\ dt \geq 0.$

The inequality follows by Theorem 5 part (a), $\underline{z}_{n+1} \geq \underline{z}_n$.

A similar argument using $\overline{z}_n$ and $\overline{z}_{n+1}$ proves that $\underline{CS}_n$ is decreasing in n.

(b) We prove that $n\overline{\pi}_n \geq (n+1)\overline{\pi}_{n+1}$. The result that $n\underline{\pi}_n$ is decreasing in n follows by a similar argument using $\overline{x}_n$. Consider the following relations:

$$\begin{aligned}
\overline{\pi}_n &= \underline{x}_n D(n\underline{x}_n) - C[D(n\underline{x}_n)] \\
&\geq [(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n] D[(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n + (n-1)\underline{x}_n] \\
&\quad -C[D[(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n + (n-1)\underline{x}_n]] \\
&\geq \left[(n+1)\underline{x}_{n+1} - \frac{(n-1)(n+1)}{n}\underline{x}_{n+1}\right] D[(n+1)\underline{x}_{n+1}] - C[D[(n+1)\underline{x}_{n+1}] \\
&= \frac{(n+1)}{n}\underline{x}_{n+1} D[(n+1)\underline{x}_{n+1}] - C[D[(n+1)\underline{x}_{n+1}] \\
&= \frac{(n+1)}{n}\left[\underline{x}_{n+1} D[(n+1)\underline{x}_{n+1}] - \frac{n}{n+1}C[D[(n+1)\underline{x}_{n+1}]\right] \\
&\geq \frac{(n+1)}{n}\left[\underline{x}_{n+1} D[(n+1)\underline{x}_{n+1}] - C[D[(n+1)\underline{x}_{n+1}]\right] \\
&= \frac{(n+1)}{n}\overline{\pi}_{n+1}.
\end{aligned}$$

The first equality follows by Lemma 11 and the first inequality by the PSNE property. The deviation $(n+1)\underline{x}_{n+1} - (n-1)\underline{x}_n$ from the equilibrium price is positive since it is equal to $\underline{z}_{n+1} - \underline{z}_n + \underline{x}_n$, and by Theorem 5 part (a), $\underline{z}_{n+1} \geq \underline{z}_n$. The second inequality follows also by Theorem 5(a), in particular using the fact that $\frac{(n+1)\underline{x}_{n+1}}{n} \geq \underline{x}_n$ (from $\underline{z}_{n+1} = (n+1)\underline{x}_{n+1} \geq \underline{z}_n = n\underline{x}_n$). The last inequality is due to $\frac{n}{n+1} < 1$.

(c) It is clear that $\overline{W}_n = \overline{CS}_n + n\overline{\pi}_n$. Then

$\overline{W}_n - \overline{W}_{n+1} = [\overline{CS}_n - \overline{CS}_{n+1}] + [n\overline{\pi}_n - (n+1)\overline{\pi}_{n+1}] \geq 0.$

The inequality follows because both terms on the RHS of the equality are positive by parts (a) and (b). A similar argument proves that $\underline{W}_n$ is decreasing in n. □

Proof of Theorem 8.

Let us consider (say) the maximal point of the equilibrium price-cost margin set.

Since $D(\cdot)$ is log-concave (log-convex), we have $\frac{D'(z)}{D(z)} \geq (\leq) \frac{D'(z')}{D(z')}$ for all $z' > z$. Thus,

$$-\frac{D(z')}{D'(z')} + \frac{D(z)}{D'(z)} \leq (\geq) 0. \tag{8}$$

Now, the first-order condition for the oligopoly with perfect complements can be written as

$$D(z_n) + m_n D'(z_n) = 0,$$

which implies that

$$m_n = -\frac{D(z_n)}{D'(z_n)}.$$

If $D(\cdot)$ is log-concave (log-convex), m_n is decreasing (increasing) in z_n, then, $\overline{m}_n = -\frac{D(\underline{z}_n)}{D'(\underline{z}_n)}$ $(\overline{m}_n = -\frac{D(\overline{z}_n)}{D'(\overline{z}_n)})$.

Thus, $\overline{m}_{n+1} - \overline{m}_n = -\frac{D(\underline{z}_{n+1})}{D'(\underline{z}_{n+1})} + \frac{D(\underline{z}_n)}{D'(\underline{z}_n)}$ $\left(\overline{m}_{n+1} - \overline{m}_n = -\frac{D(\overline{z}_{n+1})}{D'(\overline{z}_{n+1})} + \frac{D(\overline{z}_n)}{D'(\overline{z}_n)}\right)$, which is negative (positive) if $D(\cdot)$ is log-concave (log-convex), by equation 8 and Theorem 5 part (a), $\underline{z}_{n+1} \geq \underline{z}_n$ $(\overline{z}_{n+1} \geq \overline{z}_n)$. □

Proof of Proposition 9.

(a) This follows from a rather standard argument. Since the n-product monopolist can replicate whatever price vector the oligopoly can use, it is obvious that the former can achieve a higher total profit than the latter.

(b) The sum (across the n firms) of the first order conditions at a symmetric PSNE is (where z stands for total price)

$$nD(z) + zD'(z) - nC'[D(z)]D'(z) = 0. \tag{9}$$

From equation (6), the first order condition for the n-product monopolist's solution is (where z stands for total price)

$$D(z) + zD'(z) - nC'[D(z)]D'(z) = 0. \tag{10}$$

Since for any $n > 1$, the LHS of (9) is an upward shift of the LHS of (10), the extremal solutions (which are the extremal zeros of the LHS's) of (9) are higher than those of (10).

Hence, price is lower for the n-product monopolist. It follows that consumer surplus is higher with the n-product monopolist.

(c) This follows directly from (a) and (b). $\square$

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