CEE www.colmex.mx Centro de Estudios Económicos El Colegio de México, A.C.

Serie documentos de trabajo

CAPITAL ACCUMULATION GAMES

Alejandro Castañeda

DOCUMENTO DE TRABAJO

Núm. III - 1993

CAPITAL ACCUMULATION GAMES[26][27].

Alejandro Castañeda Sabido

El Colegio de México

Camino al Ajusco N.20

México, D.F. C.P 01000

Fax (5) 645-0464

Tel 645-0210 ó 645-5955 ext.4087

The paper uses numerical techniques to solve an investment game with fixed costs of adjustment in discrete time. The computational technique uses a finite dimensional representation of the value functions for the players. The finite dimensional basis used for this approximation are bilinear cardinal functions. The model studies a dynamic game in which firms have to decide how much to invest and when to do that. The firms condition their behavior on the state of the system. The firms have to pay a fixed cost each time they want to change their level of capital. Instead of studying the entry deterrence case, the model studies a game in which given the degree of interaction, fixed costs and the depreciation rate have the right size for two firms to be active in equilibrium. In equilibrium the firms will alternate in their investment behavior and will accumulate capital in such a way as to delay the investment of the opponent, so that firms are temporary Stackelberg leaders. The model could be use as a fully specified theoretical model that predicts alternation in market share. The model shows results reminiscent of the literature on the subject. The optimal response function when the firms decide to invest is higher for the noncooperative duopolists than for the collusive solution. Furthermore, other things being equal, the noncooperative firms wait less time to invest than the cooperative duopolists.

[26] I am mostly grateful to Timothy Bresnahan and Ken Judd for very helpful comments and strong support. All errors are soley mine.

[27] Preliminary, not for quotation.

INTRODUCTION.

Traditionally, investment has been modelled in the industrial organization literature as a strategic weapon. We can distinguish two approaches to modelling within this literature. In the first approach, investment is used as a tool to preclude other firms from having a large market share. The second approach is that on entry deterrence, in which the incumbent firm overinvests, installing more capacity than it would have installed, were the threat of entry absent. The increase in investment is designed to to deter the potential entrant from entering the market. Representative papers in this tradition include those by Dixit (1980), Spence (1977), Maskin and Tirole (1988 (b)) and Eaton and Lipsey (1980).

The papers by Dixit (1980) and Spence (1977), assume a once-and-for-all choice in the firm's level of capital. This assumption precludes their model from being truly dynamic. The work by Maskin and Tirole (1988 (b)) supposes an alternating move environment, in which the incumbent overinvests in each turn, in order to preclude the other firm from entering the market. Eaton and Lipsey (1980) analyze a continuous-time model, in which a monopolist invests early, in order to halt the potential rival from entering the market. Both studies exhibit rent dissipation. Firms overinvest in order to deter the potential entrant from enter the market. For Eaton and Lipsey, the additional capital has no productive abilities; it is used soley to deter entry. Therefore, the additional resources allocated to that capital are wasted. By contrast, in Maskin and Tirole, the additional capital is employed for productive uses. Assuming that firms produce at full capacity, Maskin and Tirole conclude that the additional capital is socially beneficial.

A problem with this line of research lies with the fact that in the real world we rarely observe a monopolist trying to defend its territory against a potential intruder. Instead, we find firms competing for market share.

The other tradition models investment as a device that prevents other firms from increasing their market share. In their view, firms overinvest in order to defend againts their rival's accumulation of capital. This tradition accepts explicitly the coexistence of firms in the market.

In this vein of research is the literature on irreversible investment: Spence (1979), Fudenberg and Tirole (1983). The latter authors studied a capital accumulation game with continuous investment (over time) and no depreciation[1]. Spence concludes that speed advantages or differences in the initial conditions permit a firm to become a leader and to choose a point in the follower's optimal response function that, given the initial conditions and

[1] They argue that the assumption of zero depreciation allows them to highlight the importance of commitment.

speed of investment, is closer to the Stackelberg leader position. Fudenberg and Tirole find early-stopping equilibria that are supported by the credible threat posed by firms investing until it reaches the follower's response function.

The main goal of these papers was to emphasize the importance of first-mover advantages in new industries.

Hanig (1986) studied a differential game in which firms invest continuously. He uses a linear quadratic differential game. In sharp contrast with the authors mentioned above, he allows for depreciation, and bounds the level of investment by explicitly modelling quadratic adjustment costs. He concludes that in comparison with the single Cournot Static Game, firms tend to overinvest in dynamic environments. The reason is similar to that found in Spence's model, where a firm overinvests in order to deter the other firm from attaining a high level of capacity. Since he allows for depreciation to erode capital, capital looses much of its commitment value, and, since there are no initial conditions advantages, both firms behave like Stackelberg leaders. Maskin and Tirole (1987) reached a similar conclusion in an alternating move environment.

As the brief review of the literature suggests, the analyses of investment as a strategic weapon have been unrealistic in several ways. First, in some cases only the entry deterrence case has been studied. Second, in other cases firms were permitted to vary their stock of capital costlessly at any time. Third, in some cases the models were not truly dynamic. Finally, some papers assumed a zero rate of depreciation.

I will study a dynamic game in which firms decide how much, and when to invest. Firms condition their behavior on the state of the system, which in the case of this model corresponds to the level of capital of both firms. These variables are payoff relevant as well. To formalize the idea of short run commitments, I introduce a fixed cost that firms have to pay each time they want to change their level of capital. Consequently, in equilibrium, the firms do not adjust their levels of capital every period. This property differs from the literature in which firms continuously adjust their levels of capital (although with some bound in the amount of investment) (Fudenberg and Tirole (1983) and Hanig (1986)). Furthermore, the inclusion of a fixed cost endogenizes the timing of adjustments. Also, in this modelling strategy, firms are allowed to move any time they wish. These two properties are in sharp contrast with those of the alternating move approach (Maskin and Tirole's models (1987), (1988 (b)).

Like Hanig, I will study a capital accumulation game which allows for depreciation. In contrast with Maskin and Tirole's study (1988 (b)), given the degree of interaction, in my work the fixed cost is small enough, and the rate of depreciation small enough, to allow both firms to be active in equilibrium. Instead of deterring entry, the firms accommodate each other.

An interesting question emerges in this context. Given the permanence of the competitor in the industry, under what conditions can a firm take advantage of the sluggishness in competitor's response, and delay the rival's investment in order to establish itself as a temporary Stackelberg leader? The main goal of this paper is to answer this question.

The primary contribution of this work lies in building a model where investment is a strategic weapon, and which entails more realistic assumptions regarding the competitive process[2]. The results are reminiscent of the previous literature on the subject. Instead of having entry deterrence, or permanent Stackelberg leadership, we obtain results in which firms alternate in their investment behavior, and accumulate capital in such a way as to delay the investment of the opponent, so that firms are temporary Stackelberg leaders. Because the rate of depreciation is positive, there are no first mover advantages. Rather, if we are willing to maintain the assumption that firms produce at full capacity, the model studied here could be considered a fully specified theory that explains the alternation in market share among firms.

[2] Fixed costs of adjustment, positive rate of depreciation, coexistence of few firms in the market and the analysis of a truly dynamic model.

Section 1 states the basic assumptions of the model. I argue that the one period return function involves a model in which capital affects marginal cost, and price competition is subsumed in the model.

Section 2 states the dynamic setting and presents the pair of functional equations that has to be satisfied in order for the model to be a Markov Perfect Equilibria. I also sketch the main computational approach[3]. Section 3 presents the parameter values chosen to solve the model. Section 4 states the results of the game. Because the one period return function exhibits the property of strategic substitutes, the firms will never choose to adjust their state variables at the same time. I also look at the play set, the set of states in which the firms will stay after the initial move. The set shows an area of multiplicity of equilibria in which we have two types of equilibria. If one firm invests, then it is optimal for the rival to stay put. However, if the first firm decides to stay put, then it is optimal for the rival to invest. The result is not surprising given the property of strategic substitutes that the return function exhibits.

In Section 5 I perform comparative statics analysis with respect to changes in the parameters of the model. In Section 6 I highlight an important result that appears to be persistent for all models;

---

[3] Appendix one has a more detailed explanation of the computational approach.

for some states, a firm will wait until the competitor is about to move, and in the period before, it invests in order to further delay the adjustment of the competitor. I call this phenomena "partial preemption" and the result is reminiscent of the Eaton and Lipsey's (1980) conclusion.

In section 7 I change the degree of interaction, and see how the firm's investment policy responds to these parameter changes. Given the partial preemption result, we see that for higher degrees of interaction the firms invest less frequently.

In Section 8 I make comparisons with the collusion case. I conclude that for higher degrees of interaction, the two product monopoly will adjust more frequently than the oligopolistic firms. For low degrees of interaction we get the opposite result. However, even in the latter case, firms attain higher levels of investment (each time that they decide to invest) than in the collusion solution. Firms will invest beyond the monopolist level, a result reminiscent of the former literature, and of the traditional static Cournot analysis.

Finally in section 9 I make a welfare analysis. I find that the social planner adjusts her capital more frequently than the oligopolistic firms. This is because the firms will follow policies of "partial preemption", which allow them to become temporary Stackelberg leaders, and delay the adjustment of the competitor as much as they can. We will also see that changes in the rate of depreciation affect negatively the present value of the producer surplus.

1. THE MODEL

Consider an industry with two symmetric firms with the return function expressed in terms of capital (x_1, x_2). This return function[4], $F^i(\cdot,\cdot)$ $i=1,2$, is strictly concave, so that for each level of capital of the rival firm, we have a well defined optimum for the first firm. Finally, I assume that $F^i_{12}(x_1,x_2)<0$ $i=1,2$, i.e. it exhibits the property of strategic substitutes.

Each time a firm wants to invest in new capital, it has to pay a fixed cost K, which is the same for both firms. The fixed cost will force the firms to be inactive during some time. A salient feature of this model is that it determines endogenously the time of investment by the firms, as well as, the size of the investment.

1.1 PROFIT FUNCTION.

I study a game with the following quadratic profit function:

$$(1) \qquad \Pi^1(K_1,K_2) = -(2(1+DB))K_1^2 - (2(1-Db))K_1K_2 + 1.5K_1 - cK_1$$

Following Fudenberg and Tirole (1983), I assume that the profit function is the reduced form of a process in which firms compete in prices. I assume in (1) that firms choose prices in a world of imperfect substitutes such that they produce at full capacity. Below, I address the appropriateness of this assumption. We notice immediately that $\Pi^1_{12}(\cdot,\cdot)<0$, since $0<DB<1$, i.e. the profit function has the property of strategic substitutes. DB has also the function of parameterizing the degree of interaction. The property

[4] See the next subsection.

of strategic substitutes implies that larger levels of the rival's capital, decreases the marginal profitability of capital for a firm. Given the endogeneity of the decision to invest in this model, this property should imply that firms will not adjust their capital at synchronous times. Rather, given that the rival has already invested, it may be optimal for a firm to stay put for some time. Indeed, this is one of the main results of the paper.

Equation (1), was obtained by calculating the inverse demand functions obtained by the maximization of the following utility function:

$$U(q_1, q_2) = -(q_1+q_2)^2 + v(q_1+q_2) - DB(q_1-q_2)^2 + M \tag{2}$$

Equation (2) posits a representative consumer, DB represents the degree of substitution between goods. I calculate the demand functions from the utility function in (2). To obtain equation (1) I solve for the inverse demand function and substitute it into the profit function of each firm. The reader may wonder why I use an imperfect substitute model to analyze quantity competition; indeed, the optimal response functions are continuous in the traditional static Cournot model, in which goods are perfect substitutes. For the imperfect substitute case, the social planner and the collusion solution, gives me an independent choice for the levels of capital of each of the two firms. In the perfect substitute case, the social planner and the collusion model will only give me a solution for the total level of capital of both firms (the sum of capital for both firms), leaving undeterminate the choice for each firm. In order to make comparisons with the social planner's solution and with the collusion case, I assume that the goods produced by the two levels of capacity, are imperfect substitutes: $0<DB<1$.

Doubts arise regarding the validity of equation (1); two remarks are in order in this regard. First, other authors have employed the same assumptions, (Hanig (1986), Maskin and Tirole (1988)[5]). Second, the profit function above can be considered an approximation to a model in which capacity affects the marginal cost of the firms, and two firms thus compete in prices in a world with differentiated products. As Tirole (1988) has suggested: " What we really mean by quantity competition is a choice of scale (K) that determines the firms's cost functions and thus, determines the conditions of price competition (p.218)". Given the representative consumer model above, I calculate the demand function, yielding the following profit function:

$$\Pi^1(K_1,K_2)=\left(\left(-\frac{(1+DB)}{(8DB)}\right)p_1(K_1,K_2)+\left(\frac{(1-DB)}{(8DB)}p_2(K_1,K_2)+.25\right)\right)(p_1(K_1,K_2)-f(K_1)) \tag{3}$$

If $f'(K_1)<0$ then $\Pi^1_{12}(K_1,K_2)>0$, as in equation (1). If we assume

---

[5] Following Kreps and Sheinkman (1983), Maskin and Tirole (1988 (b)) argue that the assumption that states that firms choose prices to clear capacity is reasonable if the marginal costs of investment is sufficiently large. I believe that this conjecture has to be proved in a dynamic setting. As far as I am aware, this is still an open question.

that the function $f^1(K_1)$ is linear then the profit function is quadratic. However, the function is not concave in K_1. In order to employ this profit function in my dynamic model I would need to assume a quadratic cost of adjustment in the level of investment, which is a property that **exponentially** increases the number of calculations[6]. One way to circumvent this problem is to assume that the function $f^1(K_1)$ is nonlinear with locally increasing returns, and asymptotic decreasing returns. This assumption bounds the choice K_1 in the one period return function. However, the return function would no longer be quadratic. If we were to consider the true profit function in this setting, I believe that the results would not significantly vary from my specification. I consider the profit function in equation (1) as an approximation of the true profit function in equation (3), with nonlinear $f'(K_1)$ [7].

---

[6] See chapter two.

[7] This assumption has been widely used in the macroeconomics literature. For example, Christiano (1990) and Hansen and Sargent (1990), assume that a quadratic function is a good approximation of more general nonquadratic functions.

2 VALUE FUNCTION

The firms maximize the present value of profits:

$$\begin{aligned} M^1(x_1,x_2) &= \sum_{\tau=0}^{\infty} \beta^{\tau} F^1(x_1(t),x_2(t)) - \sum_{i=1}^{\infty} \beta^{\gamma_i^1} K \\ M^2(x_1,x_2) &= \sum_{\tau=0}^{\infty} \beta^{\tau} F^2(x_1(t),x_2(t)) - \sum_{i=1}^{\infty} \beta^{\gamma_i^2} K \end{aligned} \tag{4}$$

where $x_1(t)$ and $x_2(t)$ satisfy the following difference equation:

$$\begin{aligned} x_1(t) &= x_1 + \sum_{(i:\gamma_i^1<t)} i_i^1 - td \\ x_2(t) &= x_2 + \sum_{(i:\gamma_i^2<t)} i_i^2 - td \end{aligned} \tag{5}$$

d is the rate of erosion of the state, γ_i^1 and γ_i^2 are the times at which firm one and two change their state variables, and the amount of the adjustment is given by i_i^1 and i_i^2 . The difference equations above express the state at time t , $x_1(t)$ and $x_2(t)$, as a function of the initial conditions plus the adjustments made by the firms up to time t , less the erosion caused by the rate of depreciation td .

Within each period, firms make two sequential decisions. At the beginning of the period, firms observe their current state. Then, they decide whether they want to pay the fixed cost or not. They both observe the decision made by both firms with regard to paying the fixed cost. Then they decide the size of the adjustment. At the end of the period, the exogenous process erodes their state variables.

For each period, firms play a two stage game. This game's extensive form is depicted in Graph 2.1. and the corresponding normal form is expressed in Graph 2.2. As we can see from the extensive form, there may be mixed strategies involved in the decision of whether to pay the fixed cost or not. The p in the following functional equation (10) , represents the probability that firm one pays the fixed cost in that period. q represents the probability that firm two pays the fixed cost. S^1 in (10) is the optimal response function of firm one. Due to the linearity of the cost adjustment function, it is function of the current state for firm two (x_2) if two does not move. If two does move, then S^1 corresponds to the intersection of the optimal response functions. A similar explanation holds for firm two.

Let $V^i(x_1,x_2)$ $i=1,2$ be the value function associated with a Markov Perfect Equilibria solution, starting at time 0 with capital (x_1,x_2) . The value function of both players is defined by the simultaneous solution of the following pair of functional equations:

$$V^1(x_1,x_2)=\underset{p,}{Max}\,p[q\underset{S^1}{Max}[F^1(S^1,S^2)+BV^1(S^1-d,S^2-d)-c_m(S^1-x_1)]$$
$$+(1-q)\underset{S^1}{Max}[F^1(S^1,x_2)+BV^1(S^1-d,x_2-d)-c_m(S^1-x_1)]-K]$$
$$+(1-p)\ [q[F^1(x_1,S^2)+BV^1(x_1-d,S^2-d)]+$$
$$(1-q)\ [F^1(x_1,x_2)+BV^1(x_1-d,x_2-d)]]=TV^1$$

(6)

$$V^2(x_1,x_2)=\underset{q,}{Max}\,q[p\underset{S^2}{Max}[F^2(S^1,S^2)+BV^2(S^1-d,S^2-d)-c_m(S^2-x_2)]$$
$$+(1-p)\underset{S^2}{Max}[F^2(x_1,S^2)+BV^2(x_1-d,S^2-d)-c_m(S^2-x_2)]-K]$$
$$+(1-q)\ [p[F^2(S^1,x_2)+BV^2(S^1-d,x_2-d)]+$$
$$(1-p)\ [F^2(x_1,x_2)+BV^2(x_1-d,x_2-d)]]=TV^2$$

A Feedback Nash equilibria is a quadruple (p,S^1,q,S^2), so that both equations in (6) are satisfied.

I define Ω_c as the set of x_1 and x_2 in which it is an equilibria that neither of the firms wants to move, p and q are equal to zero in (6). Alternatively, Ω_1 is the set of states in which only firm one wants to move, p is equal to one and q is equal to zero in (6) above. When p is equal to zero and q is equal to one then the firms are in the set Ω_2. If p is equal to one and q is equal to one, then the firms are in the trigger set for both firms Ω_m, whenever firms are in this set they move to the intersection of the optimal response functions $S^i(x_j)$. Finally, I obtain the set in which p and q are between one and zero. This set contains mixed strategy equilibrium. Let me call this set Ω_{mx}. $s^1(x_2)$ represents the low boundary between the trigger set for firm one and the continuation set when the state of firm two is at x_2 [8] . Finally, s represents the point at which firm one (and firm two) decides to move whenever both firms are moving together.

If the extensive form of the game for each period is not defined in this way, every time in which there is not a consistent outcome in pure strategies, I would have had a very difficult task in calculating the mixed strategy equilibria. This is so, because the action space is continuous[9]. The fact that firms observe whether they have paid the fixed cost in the first stage of the period, allows me to restrict the strategies upon which firms can randomize. In this setting, the mixed strategy equilibria occurs only when firms have not a consistent decision in pure strategies about paying a fixed cost.

We can view the right hand side of the equations in (6) , as an operator which maps tomorrow's value function into today's value function. **If the solution exists,** (see the paragraph below) the standard procedure for solving the pair of functional equations expressed above is to start with an arbitrary function and to iterate in the following map until convergence is reached.

---

[8] Since capital can be reduced and increased, there is a high boundary between the continuation set and the trigger set for each of the two firms. But, because depreciation is a one sided process these will never be attained after the second movement.

[9] Even in the case of discrete action spaces it is impossible to calculate the mixed strategy equilibria in the computer, whenever firms are mixing between several possible choices.

(7) $$V^{i,n} = TV^{i,n-1} \quad i=1,2$$

In the single decision case with standard assumptions on the one period return function $F^i(\cdot,\cdot)$ and the discount factor B, the contraction mapping theorem guarantees the convergence of the iteration in (7) . Obviously, existence for strategic environments is not a simple extrapolation of this argument.

I am not aware of an existence proof for this type of games. Dutta and Rustichini (Dutta and Rustichini 1991) have proved existence in a game similar to this one, but with one state variable[10] [11]. We have existence theorems for games with discrete state space and controls restricted to points in the grid (Fudenberg and Tirole (1991), chapter 13 theorem 13.2).

Existence from a computational perspective is verified by seeing whether the approximation in the computer is actually converging. As Judd (Judd 1990 (b)) has suggested, the computational approach to these models may be viewed as an analogous procedure towards

[10] Since the optimal response function is non-convex, i.e. for some states the player does not move and for others she does. Traditional existence results which rely on the Kakutani Fixed Point Theorem cannot be used. The Dutta and Rustichini's method uses some kind of monotonicity in the optimal response function, and then, they use Tarski fixed point theorem to prove existence. I suspect that this approach may work for this model.

[11] Existence of the open loop equilibria is no problem. Once the other firms strategy is chosen, the optimization problem becomes a single agent procedure. In continuous time, the proof of a existence of an optimal policy for a single decision maker would apply (Bensoussan and Crouhy (1983)).

proving existence. The computational solution may be thought of as an epsilon equilibria of the original problem.

Given the fact that I solve the model in the computer in discrete time, differentiability cannot be guaranted for the whole state space. In particular, the value function is not differentiable at the boundary between the continuation set and the trigger set. The fact that a firm can either decide to wait for a period to adjust, or instead adjust right now, implies two different control choices for that firm. It can wait one period to exert its control, thereby choosing implicitly a discretely lower state variable for tomorrow. Alternatively, it can exert its control without delay and choose a higher value of the state variable for tomorrow. Clearly, this implies that the value function is not differentiable at any boundary between the continuation set and any of the trigger sets. Moreover, for all initial states in the continuation set that hit the boundary exactly, the value function remains nondifferentiable[12]. This last reasoning insinuates that the value function is only piecewise differentiable in discrete time. Consequently, it is important to keep our approximation in a space suitable for these properties of the value function. A similar proof to the one made in Castañeda (1992 (a)) in continuous time, will show that the value function is continuous in discrete time.

[12] See Lucas and Stokey (1990) page 118 for an example that shows this particular problem.

In appendix one I explain more carefully the computational approach, here, I only sketch the main procedure. Given the theoretical restrictions analyzed so far, it is natural to look for value functions in the space $\hat{B}$ of piecewise differentiable functions mapping $D\subset\Re^2$ into $\Re^1$. Since the computer cannot approximate the whole space of piecewise differentiable functions, I look for a finite dimensional representation of the value function [13].

I calculate the mapping T in (7) in two stages. First, I solve for the Nash equilibria in a square lattice with equally distant points in both dimensions so that T is exactly satisfied in (7) for any point in the lattice. Any point in the lattice is given by the following ordered pair: (x_i,x_j). Where $x_i=c_0^1+ih$ $i=0,99$ and $x_j=c_0^2+jh$ $j=0,99$. Furthermore, the origin of the lattice is on the forty five degree line so that $c_0^1=c_0^2$ [14]. Second, I seek for an interpolation method that best satisfies the theoretical restrictions of the model, and that better summarizes the information obtained from the points of the lattice. By using this procedure, I replace the theoretical mapping from the space of continuous functions into continuous functions represented in (7) by T, into a finite dimensional approximation of that map.

---

[13] Judd (1990) is a very good paper on computational approaches in economic analysis. The impact of that paper in this part of this chapter is considerable.

[14] In general $c_0^1=c_0^2$ were very close to zero, so that the whole grid was in the positive orthant.

In the interpolating procedure, I choose a finite dimensional basis and represent the approximated value function in this subspace. Then, through the iteration procedure, I map this finite dimensional approximation into another finite dimensional approximation until I reach a level in which V^n is reasonably similar to V^{n+1}.

Given the fact that the value function is piecewise differentiable, it appears that the best approach is to use a finite element basis with small support. With a small support basis, errors in approximation in one part of the state do not affect the interpolation in another part of the state. In finite element approaches the interpolation proceeds locally, subinterval by subinterval. The global approximation is obtained by patching together all the subintervals. The approximated Value Function is then expressed in the following way:

$$(8) \qquad V^i(x_1,x_2)=\sum_{j=0}^{n} a_j\phi_j(x_1,x_2) \qquad i=1,2$$

A suitable basis to implement the above procedure is the use of bilinear cardinal functions. These cardinal functions span the space of purely continuous functions C^0 when the size of the grid tends to zero (the number of points go up to infinity) (Lancaster and Salkauskas (1986)). I use this to gain computational speed.

Bilinear cardinal functions are easy to construct[15].

[15] This argument is not true for higher order approximations. In such case the cardinal functions may have to be calculated. And when we calculate them, they may not have the small support property.

3 CALIBRATION

In Table 1 I show the values chosen for each possible parameter: The discount factor, the size of the fixed cost, the rate of depreciation, and the degree of strategic interaction.

The rates of depreciation are monthly depreciation rates that correspond to a yearly depreciation rate of 15 percen and 25 percent. Firms revise their investment decisions every month, and the annual rate of interest is 10 percent. I choose 0.4 for the marginal cost of production. The marginal cost for adding capacity is 0.25. The fixed costs of adjustment are .34, .5 and .67. The degree of substitution can take on four levels, 0.58, 0.6, 0.7 and .85[16]. The profit income ratio for those levels of interaction at the static Cournot Nash Equilibria are 0.63, 0.63, 0.65. and 0.68 respectively. These numbers may appear too high, the reason for obtaining them is that v in equation (2) is very large, which yields a very high constant term in the linear demand functions for each firm. If v were not large, the dynamics in the game would have been very close to the axes of the positive orthant, which would not permit me to accurately characterize the game. Table 1 summarizes this information. By combining all of the parameter values, there is a total of 36 models. Finally, there is a last important property, it is more costly to reduce capital than to increase it. Hence, if firms want to reduce their

[16] I tried other values for the degree of substitution (*DB*) , .3 and .5. I was not able to get convergence for some models with these parameters values, the value function was cycling. This evidence is consistent with other people working in this field.

level of capital, they will have to pay an extra fixed cost equal to 0.25.

Table 1

Discount Factor B	.992
Fixed Cost K	.34, .5, .67
Depreciation D	0.012, 0.019
Degree of Substitution DB	0.58, 0.6, 0.65, 0.7, 0.75, 0.85
Marginal Cost of Investment cm	0.25
Marginal Cost	0.4

4 COMPUTATIONAL RESULTS.

4.1 SYNCHRONIZATION.

In Castañeda (1992 (b)) is argued that synchronization is by far the most robust outcome of that work's simulations. I select one hundred equidistant initial conditions for each model and then run 10,000 periods in a simulation for each initial condition. In this process, I check whether synchronization was the resulting asymptotic outcome. I also checked whether there was a significant difference in the level of capital that both firms would hold, as compared with the size of the inaction set[17]. One of the most important changes that I make in passing from Castañeda (1992 (b)) to this work is the switch from strategic complementarities to strategic substitutes in the one period return function $\Pi(\cdot,\cdot)$. This change produces completely different results with respect to the simulations. For all models and for all initial conditions **non-synchronization was the only outcome.** In all cases, the difference between the level of capital was significant, when compared with the size of the set of inaction. Additionaly, synchronization in capital adjustment is an unstable process. Firms would rather invest in new capital in asynchronous times. Villas-Boas (1990) obtains a similar result in a model in which the strategy space is considerably reduced. The intuition for this result comes from the assumption that the profit function has the property of strategic substitutes. We remember that in the static

[17] The continuation set (Ω_c) in the terminology of chapter two.

Cournot case with strategic substitutes, the optimal response function is negatively sloped. The higher the rival's quantity (capital), the lower the quantity (capital) that the firm wants to have. In this model, the intuition of the static Cournot model is translated into the following: If the rival wants to have a higher level of capital, it is optimal for the first firm to maintain its level of capital at a lower level. In the next section we will see that there is an area of the state space in which it is optimal for a firm to invest if the rival wants to stay put, but would rather stay put if the rival wants to invest. The reason for the existence of this area follows the same intuition. Therefore, if the rival firm wants to invest now, it is optimal for the first firm to stay put and to maintain a low level of capital.

The inclusion of a fixed cost of adjustment introduces a a new dimension in the analysis of capital accumulation games. In Maskin and Tirole's model (1987), as well as in Hanig's model (1986), symmetric firms maintain the same high level of capital in the steady state (i.e. both firms behave like Stackelberg leaders). When we include a fixed cost in the analysis, the equilibrium behavior changes radically. Both firms alternate their levels of capital and the levels of capital are never equal between the firms.

4.2 PLAY SET[18].

Graph one illustrates the different sets for the state space. In all models I noticed that there is a large set of states that yield multiple equilibria. Firm one will move only if the other firm does not move, and will stay put if the other firm moves. Given the method of solution proposed above-i.e. to flip a coin[19]-, the firm that moves adjusts its capital up to the optimal response function. This result is not surprising, given the assumption of strategic substitutes. Both firms seek to preempt the rival, but if one randomly gets to move first, then the other firm would rather stay put. As mentioned in the last section, the intuition for this result follows from the slope of the optimal response function of the static game with strategic substitutes. The higher the rival's capital, the lower the level of capital that a firm wants to have. In the context of this model, the intuition is as follows: If the rival wants to invest and therefore have a higher level of capital, the firm would prefer to maintain a low level of

---

[18] We can define the play set as the set of states in which the game will be played after the second move. Some of these states may be visited only very few times, whereas others will be frequently visited. By looking at the shape of the state and the simulation results, it seems that for the case with strategic substitutes firms will visit more states more often than in the case with strategic complementarities (see Castañeda (1992 (b)). The reason for this result, is that synchronization is locally stable for the strategic complements case, whereas is locally unstable for the strategic substitutes case.

[19] By flipping a coin, I capture idiosyncratic shocks to any firm that permits the other to preempt it.

capital. This result implies that synchronization is locally unstable. For reference I will call this set Ω_{mg} .

If we look at graphs one and two we will see that Ω_{mg} is far larger for higher degrees of interaction (graph one corresponds to $DB=.50$, and graph two corresponds to $DB=.85$). Given the fact that firms are strategic substitutes, for higher degrees of interaction there are more states in which both firms seek to preempt the rival. Any negative exogenous shock that affects a firm will prove advantageous for the other firm, since the shock allows the firm to invest, and preempt, at least temporarily, the rival.

It is interesting to analyze the shape of a typical response function $s^i(x_j)$ for a firm that decides to move. For states close to the trigger set of the other firm Ω_j , the optimal response function is increasing. This is because the firm that moves (i) is aware that the rival will move, within a short period. Given the fact that capital is a strategic substitute variable, it would rather move to a lower level, the closer the time for the rival (j) to move. As we get further away from the trigger set of the other firm Ω_j , the optimal response function $s^i(x_j)$ becomes negatively sloped as in the static Cournot case. The reason is the traditional Cournot explanation: the higher the other's firm capital, the lower the new level of capital that the firm that is moving wants to have. These properties of the optimal response function appear to be valid for all models. Again, as we shift from a higher degree of interaction (graph one), to a lower degree (graph two), the optimal response function moves down. If the degree of interaction is high, the firms will behave more like Stackelberg leaders whenever they decide to move. The higher the degree of interaction, the higher the level of investment that the firms want to maintain.

5 COMPARATIVE STATICS

5.1 INCREASE IN THE FIXED COST

The size of the play set increases for both levels of depreciation. Further, $S^i(x_j)$ the optimal response function goes up, and $s^i(x_j)$ the boundary between the trigger set and the continuation set for each firm goes down, this property holds true for both levels of depreciation. Graph three illustrates these results.

Regarding simulations[20], I obtain the following results: The frequency of capital adjustment decreases as the size of the fixed costs increases. This happens for both levels of depreciation, for all values of DB, and for all the changes between $K=.34$ and $K=.5$, and between $K=.5$ and $K=.67$ (See Table 2).

5.2 INCREASE IN THE RATE OF DEPRECIATION

When the rate of depreciation increases the size of the Play Set goes up for all values of DB. $S^i(x_j)$, the optimal response function, goes up as the rate of depreciation goes up for all values of the fixed cost, and for all values of degree of interaction. $s^i(x_j)$, the boundary between the trigger set for any of the firms Ω_i and the continuation set Ω_c drops, again this is

[20] As I explain below, for each initial condition and for each model in the simulations, I count the number of adjustments in the level of capital that the firms make. This measure gives me an idea on how frequent the firms adjust capital.

true for all fixed costs, and for all values taken on by the degree of interaction. Graph three illustrates these results.

For the three levels of the fixed cost $K=.34$, $K=.5$ and $K=.67$ and for all levels of the degree of interaction DB , we find that an in the rate of depreciation increases the frequency of capital adjustments (See Table 2).

6 PARTIAL PREEMPTION

It is worth highlighting a result that appears to be persistent for all models. If we look closely at Graph one we will notice that, in the boundary between the continuation set and the set of multiple equilibria Ω_{me} (the small white squares), there are states in which only one firm moves. The result is reminiscent of the literature on Preemption by Maskin and Tirole (1988 (b)) and Eaton and Lipsey (1980). A firm adjusts its capital just before the other firm is about to be interested in moving, and in doing so, adjusts to the highest level of capital that it is possible in the game. This tactic allows the firm to succeed as a temporary "Stackelberg" leader, at least until the other firm reaches the boundary between its trigger set and the continuation set. Given the property of strategic substitutes, the firm adjusting its capital delays the increase in the capital of the other firm until the firm that was preempted has such a low level of capital that it decides to increase. If we look carefully at graphs one and two, we notice that the area of preemption is larger for higher degrees of interaction. This happens because the area of multiplicity Ω_{me}, is larger for higher degrees of interaction. Furthermore, for higher degrees of interaction, the area of preemption is located at higher levels of the state than for lower degrees. The intuition here is straightforward, since for higher degrees of interaction, firms will wait less time to preempt their rival.

7 CHANGES IN THE DEGREE OF INTERACTION

In Table 2 I report the average number of adjustments in the level of capital made by a firm. The method of calculating these numbers is as follows. As stated in section 4.2.1 I selected one hundred equidistant initial condition for each parameter model. Then I simulated 10,000 periods for each initial condition and for each parameter model. In this process I count the number of adjustments that the firms make. The numbers indicate an average over the one hundred initial conditions for each set of parameter values[21]. We note that, as the degree of interaction increases, the average number of adjustments decreases. The reasons for this are twofold. On the one hand, by looking at graphs one and two, we note that the optimal response function is higher for higher degrees of

[21] The standard deviation of the average number of adjustments was in all cases extremely small, so that the average is a good indicator of the number of adjustments for each model.

interaction. Caeteris paribus, this fact implies more time between adjustments. At the same time, the boundaries between the trigger sets for both degrees of interaction are almost at the same level. The second reason comes from the idea of "partial preemption". In graph one we see (using the arrows), that firms delay the competitor's adjustment by moving the period before the other firm is willing to move. If we compare graph one with graph two we see that the policy of partial preemption delays to a lesser degree the competitor's adjustment for low degrees of interaction. Also, the fact that depreciation is an exponential process forces the state to move to the area of multiple equilibria (Ω_{me}). This allows the firm to follow policies of partial preemption more often.

8 COMPARISON WITH COLLUSION

In this section I make comparisons between the collusive case and the noncooperative solution. As it has been argued previously, I assume a world of imperfect substitutes, hence, the solution for the collusion case is well defined.

Graph one and graph four illustrate the play set for the game solution and for the collusion solution. We can see that, as in Hanig (1986), Maskin and Tirole (1987), and as in the static Cournot case, firms overinvest in comparison with the monopoly solution. We see this in the graphs, where the optimal response function for both firms ($S^i(x_j)$ $i=1,2$ $i\neq j$) is higher than the optimal response function for the monopoly case. For any of the rival's states, a firm will invest to a higher level than the level it would have chosen were collusion feasible. There is an additional dimension, due to the presence of the fixed cost of adjustment[22]. The boundary ($s^i(x_j)$ $i=1,2$ $i \neq j$) at which the firms adjust is higher for the game solution than the boundary at which the monopoly decides to adjust. If firms are not cooperating, then, other things being equal they will decide to invest in new capital before the time they would decide to invest if they were fully cooperating.

These two differences in the solution between the collusion case and the game solution emerge from a negative externality between the firms for the game solution. When choosing to invest with respect to both the timing and the amount, the individual firm takes into account the effect of its two decisions on its own profits, and not the effect on the industry profits. Hence, each firm chooses a higher level of investment and will invest earlier than would be optimally desirable from the point of view of the entire market (the collusion solution).

Table 2 expresses the average number of adjustments obtained from

[22] This dimension was absent in the former literature, because they did not include fixed cost of adjustment.

the simulations[23]. The results, furnish me with an idea of how the frequency of capital adjustment varies between the game case and the collusion case. This same measure was used in the analysis above. In contrast with the strategic complements case (Castañeda (1992 (b))), the frequency of capital adjustment is higher under the collusion solution than under the game solution for a degree of interaction (DB) , equal to .58,.6, and .65. This is due to the negative effect of the competitors's strategic variable (capital) on the firm's desired level of capital.

Any time the firm moves, it tries to further delay the move of the other firm in such a way that it can maintain "Stackelberg" leadership for that period of time. This performance by both firms reduces the frequency of capital adjustments relatively to the solution for the collusion case. Graphs one and four illuminate this result. Note that the optimal response function for the game solution is higher than the optimal response for the collusion case. Secondly, due to the fact that depreciation is an exponential process, depreciation tends to move the state towards the area of partial preemption. As stated in the last section, the policies of partial preemption have a strong effect in delaying the adjustment of the rival for these degrees of interaction.

---

[23] The procedure for calculating those numbers was stated in the last section. I followed an identiacal procedure for both the game and the collusion case.

As we increase the fixed cost, we also see in Table 2 that, for a degree of substitution less than or equal to .65, the difference in the average number of adjustments between the collusion case and the game decreases. All else being equal, at higher levels of fixed costs, the monopolists will try to economize. Hence, if fixed costs are large, the monopolist will reduce the number of adjustments at a faster rate than the noncooperative duopolists do. This phenomena closes the gap in the frequency of capital adjustments between the noncooperative duopolists and the monopolist.

The results when (DB) equals .7,.75 and .85 are different. First, we note that the frequency of capital adjustment is higher for the game case than for the collusion case, although this difference is small. By looking at graph one and two, we see that the area of partial preemption is much smaller for low degrees of interaction than for higher degrees. Furthermore, as noted above, the impact of delaying the competitor's adjustment by preempting is much smaller for low degrees of interaction than for larger degrees.

TABLE 2
COMPARISON IN THE NUMBER OF ADJUSTMENTS BETWEEN THE GAME, AND THE COLLUSION CASE.

B	FIXED COST	DEPRE	*DB*	AVERAGE NUMBER OF ADJUST. COLL.	AVERAGE NUMBER OF ADJUST. GAME	DIFFERENCE IN ADJUST BETWEEN COLL. AND GAME
.996	.34	.012	.58	143	139	4
			.6	147	140	7
			.65	149	144	5
			.7	150	152	-2
			.75	156	159	-3
			.85	157	163	-6
		.019	.58	193	182	11
			.6	196	186	10
			.65	200	196	4
			.7	200	200	0
			.75	208	204	4
			.85	208	217	-9
	.5	.012	.58	127	122	5
			.6	128	125	3
			.65	128	128	0
			.7	131	133	-2
			.75	133	135	-2
			.85	137	139	-2
		.019	.58	169	161	8
			.6	169	166	3
			.65	172	172	0
			.7	175	175	0
			.75	178	182	-4
			.85	182	185	-3

CONTINUATION TABLE 2
COMPARISON IN THE NUMBER OF ADJUSTMENTS BETWEEN THE GAME, AND THE COLLUSION CASE.

B	FIXED COST	DEPRE	*DB*	AVERAGE NUMBER OF ADJUST. COLL.	AVERAGE NUMBER OF ADJUST. GAME	DIFFERENCE IN ADJUST BETWEEN COLL. AND GAME
.996	.67	.012	.58	112	112	0
			.6	112	111	1
			.65	116	116	0
			.7	116	118	-2
			.75	119	122	-3
			.85	121	125	-4
		.019	.58	150	149	1
			.6	151	149	1
			.65	153	153	0
			.7	154	158	-4
			.75	156	161	-5
			.85	162	164	-2

Note: The figures in the table above were calculated as an average of the one hundred initial conditions. The standard deviation, was in all cases very small.

9 WELFARE ANALYSIS.

For all models, the frequency of capital adjustment is higher for the social planner than for the game case. When we compare the social planner case with the game case, we must take into account the fact that the social planner wants to maintain, on average, the stock of capital at a higher level than the noncooperative duopolists. Since depreciation is an exponential process[24], it erodes capital at a faster rate for higher levels of capital. This result implies a higher rate of adjustment for the social planner. Secondly, the externality mentioned before, in which both firms adjust capital in such a way to maintain the rival's capital as low as possible, delays the rate of adjustment of capital of the noncooperative duopolists when we compare it to the social planner's rate of adjustment.

In table 3, I calculate the changes in the producer surplus resulting from a change in the rate of depreciation[25]. A 50 percent increase in the rate of depreciation generates a decrease in the producer surplus of approximately 4 to 7 percent. We note that as the rate of interaction goes down, the decrease in the producer surplus is larger. This conclusion is not surprising in

[24] In chapter three, the depreciation effect affected the social planner in the opposite direction. This was so, because the social planner wanted to maintain on average lower prices than the noncooperative duopolists.

[25] In some countries the governments can increase the rate of depreciation artificially, by changing the tax policy.

view of the results found above. As the degree of interaction decreases, the firms preempt less, and therefore increase the frequency of investment. Consequently, we conclude that in industries that are almost monopolies (i.e. those that exhibit lower degrees of interaction), a higher rate of depreciation has a greater impact on producer surplus than in more oligopolistic industries.

TABLE 3 AVERAGE CHANGES IN PRODUCER SURPLUS

AFTER A CHANGE IN THE RATE OF DEPRECIATION.

B	FIXED COST	*DB*	MEAN RATIO OF PRODUCER SURPLUS	STANDARD DEVIATION
.996	.34	.58	0.961	0.000079
		.65	0.960	0.000066
		.75	0.957	0.000055
		.85	0.956	0.000046
	.5	.58	0.950	0.000117
		.65	0.949	0.000099
		.75	0.945	0.000083
		.85	0.943	0.000073
	.67	.58	0.940	0.000159
		.65	0.937	0.000138
		.75	0.934	0.000117
		.85	0.931	0.000105

The figures above, were calculated as an average of the individual ratios of producer surplus for each of the 10000 nodes, when we increase the rate of depreciation from 0.012 to 0.019. The numerator in the ratio for each node, corresponds to the producer surplus after the change.

CONCLUDING REMARKS.

I have studyied a capital accumulation game, where fixed costs are large enough to warrant inaction for some states for both firms, but small enough to warrant the accommodation of both firms in the market.

My results are reminiscent of the literature on preemption. One firm will wait until the other firm is about to move. Then, in the instant before the competitor is willing to move, the firm adjusts its level of capital to the highest possible level achieved in the game. This further delays the adjustment of the rival in its level of capital. This behavior appeared consistently in all parameter models considered in this work. However, the impact of these policies appears to be larger for larger degrees of interaction.

The introduction of fixed adjustment costs provides a new dimension in the analysis on quantity competition. In contrast with the former literature on the subject, where Hanig (1986) and Maskin and Tirole (1987), showed that, in equilibrium, symmetric firms will maintain the same level of capital above the collusion level, my research shows that, when we bring fixed adjustment costs into the analysis, the firms will not keep the same level of capital all the time. Rather, they will alternate in the level of of capital. Furthermore, if we accept the assumption that firms choose prices in such a way that they always produce at full capacity. This model is a fully specified theory, that predicts alternations in market share. This prediction was necessarily absent in the earlier literature.

When I compare the social planner's solution with the game's solution, I conclude that the social planner adjusts her variables more frequently relative to oligopolistic firms. This result stems from the fact that firms behave like "Stackelberg" leaders, and consequently try to delay the adjustments of the competitor. Secondly, since depreciation affects capital at an exponential rate, and the social planner wants to maintain a higher level of capital on average, depreciation further enhances the "Stackelberg" effect.

With regard to the collusion case, I obtain mixed results. For a high degree of interaction $(DB \leq .65)$, the two-product monopoly (collusion) adjusts more frequently than the oligopolistic firms. This result follows from the fact that both firms try to behave like "Stackelberg" leaders, thereby delaying the adjustment in the competitor's capital for as long as they can. They do so by adjusting right before the other firm wants to move. At the same time, the fact that depreciation is an exponential process forces the state to move towards the area of partial preemption. For lower degrees of interaction $(DB > .65)$, the collusive outcome adjust less frequently than the oligopoly. For this case, the area of partial preemption is much smaller, and its effect in terms of delaying the adjustment of the rival is less important. The Stackelberg effect is not strong enough to reverse the fact that the monopolist internalizes the costs of adjustment.

When we analyze the shape of the play set for the game case and the monopoly case, we note that the optimal response function for both firms is higher for the noncooperative duopolists than for the collusive solution. Furthermore, the boundary between the trigger set for each firm and the continuation set, is located at a higher level in the game solution than in the collusion case. All else being equal, firms wait less time to adjust in noncooperative environments than in the fully cooperative one.

Appendix 1.

As mentioned in the main text the computational approach works in two stages. First I solve for the Nash equilibria in a square lattice with equally distant points in both dimensions so that T in equation (1) is exactly satisfied for any point in the lattice. Second, I look for an interpolation method that satisfies the piecewise differentiability imposed by the discretization of time. In this second procedure I choose a finite dimensional basis and represent our approximated value function in this subspace. Then through the iteration procedure I map this finite dimensional approximation into another finite dimensioanl approximation until I reach a level in which V^n is reasonably similar to V^{n+1}. As mentioned in the tex I use bilinear cardinal function because they span the space of purely continuous functions C^0 when the size of the grid tends to zero (the number of points go up to infinity), and therefore satisfies the property of piecewise differentiability of the theoretical value function. Secondly, this basis gives me computational speed. By definition of the bilinear cardinal functions, the coefficients which accompany this basis in the representation of the value function, equation (13), are just the value function calculated in the solution of the Nash Equilibria at each node surrounding the point of interest for evaluating the function. In other words, the difficult task is to calculate the cardinal functions. Once these are calculated the projection coefficients (the a_j in (13)) are trivially determined.

Given the properties of this model, it is not advisable to make use of the spectral methods that have been employed in economics to approximate the value function (See Judd 1990 (b)). The use of Chebychev polynomials for example, imposes a degree of smoothness in the solution (C^∞) which is clearly undesirable for this problem.

As mentioned already, the interpolation function is found by looking in the space of bilinear functions for a function that best interpolates the value function in each of the squares. The next step is to patch together all of the small approximations and get a global approximation of the value function. The result will be a C^0 function.

More explicitly, following the technique of finite element, for each subsquare $[x_1^j, x_1^{j+1}]X[x_2^k, x_2^{k+1}]$, I use the information gotten from the first stage at each one of the nodes of this subsquare[28]. I search for a bilinear function[29] that best approximates the value function in this subsquare. In practice this is usually done by the use of the standard rectangle. The rectangle with vertices in the points (1,1), (-1,1), (-1,-1), (1,-1). I map the value of x_1

---

[28] In the context of this model, this information comprises only the values of the operator $TV^{i,j}$ at each of the nodes.

[29] The class of bilinear functions is spanned by the monomials 1, x, y and xy.

and x_2 to the standard rectangle by the following functions:

$$\zeta=1-\frac{2\,(x_1^{j+1}-x_1)}{x_1^{j+1}-x_1}$$
$$\eta=1-\frac{2\,(x_2^{j+1}-x_2)}{x_2^{j+1}-x_2}$$

The cardinal functions are given by the following expression:

$$\phi_1(\zeta,\eta)=\frac{1}{4}(1+\zeta+\eta+\zeta\eta)$$
$$\phi_2(\zeta,\eta)=\frac{1}{4}(1-\zeta-\eta+\zeta\eta)$$
$$\phi_3(\zeta,\eta)=\frac{1}{4}(1-\zeta+\eta-\zeta\eta)$$
$$\phi_4(\zeta,\eta)=\frac{1}{4}(1+\zeta-\eta-\zeta\eta)$$

We notice immediately that $\phi_1=1$ when $\zeta=1$ and $\eta=1$. $\phi_2=1$ when $\zeta=-1$ and $\eta=1$. $\phi_3=1$ when $\zeta=-1$ and $\eta=-1$. Finally $\phi_4=1$, when $\zeta=1$ and $\eta=-1$. The interpolant to the data $TV^{1,j}(x_1^{j+1},x_2^{k+1})$, $TV^{1,j}(x_1^{j},x_2^{k+1})$, $TV^{1,j}(x_1^{j},x_2^{k})$ and $TV^{1,j}(x_1^{j+1},x_2^{k})$ is given by the following equation:

$$B(\zeta,\eta)=TV^{1,j}(x_1^{i+1},x_2^{k+1})\,\phi_1(\zeta,\eta)+TV^{1,j}(x_1^{i},x_2^{k+1})\,\phi_2(\zeta,\eta)$$
$$+TV^{1,j}(x_1^{i},x_2^{k})\,\phi_3(\zeta,\eta)+TV^{1,j}(x_1^{i+1},x_2^{k})\,\phi_3(\zeta,\eta)$$

Given the linearity of the approach, in addition to the fact that the axes are parallel to the subsquares, there is a unique surface that interpolates the whole grid (see Lancaster and Salkauskas 1986).

As mentioned in section 2.1.3, an important property of this model is that the optimal response function of firm i , S^i , is independent of the state in which the firm is located. It only depends on the other firm's state. This property holds because the cost of adjustment function of the state is linear. This characteristic allows me to speed up the calculations in the computer. Since I am assuming that the operator $TV^{1,j}(\cdot,\cdot)$ is C^0 , in order to calculate the optimal response function $S_1^*(\cdot)$ whenever firm one is moving I use a golden section algorithm. While one shortcoming of this algorithm is its speed, it is the only algorithm available for C^0 functions.

It is interesting to point out that the uniqueness of the interpolating polynomial is not maintained when I try to approximate the surface with polynomials of higher degree. For instance, if I use bicubic splines, some conditions on the second derivative have to be imposed. Additionally, for polynomials of

higher degrees, the cardinal functions do not possess the small support property. In the linear case, the rational for using the cardinal functions is that they possess the same nice properties as the B splines functions. In higher degrees, the Cardinal functions may not be easy to construct. The calculation may involve the inversion of a Vandermondian matrix, which we know is a dangerous procedure (see Lancaster and Salkauskas (1986)).
Consequently, if I want to extend the interpolation method to polynomials of second degree,- and hence allow for a higher degree of smoothness in the value function-, I should use B splines, since they maintain the small support property and permit a more efficient the approximation.

BIBLIOGRAPHY.

Christiano, L.J. (1990): "Solving the Stochastic Growth Model by Linear Quadratic Approximation and Value Function Iteration" *Journal of Business and Economic Statistics*, January 1990.

Dixit, A. (1980): "The Role of Investment in Entry-Deterrence," *The Economic Journal*, 90, 95-106.

Dutta, P. and A. Rustichini (1990): "s-S Equilibria in Stochastic Games with an Application to Product Innovations," mimeo. Paper Presented in the S.I.T.E. seminar at Stanford University summer 1991.

Eaton, B.C. and R.G. Lipsey (1980): "Exit Barriers and Entry Barriers, the Durability of Capital as a Barrier to Entry," *Bell Journal of Economics*, 12, 593-604.

Fudenberg, D. and J. Tirole (1983): "Capital as Commitment: Strategic Investment to Deter Mobility," *Journal of Economic Theory,* 31, 227-250.

Fudenberg, D. and J. Tirole (1991): *Game Theory*. Cambridge: MIT Press.

Hanig, M. (1986): *Differential Gaming Models of Oligopoly*. Ph.D. thesis, Department of Economics, Massachusetts Institute of Technology.

Hansen, L.P. and T.J. Sargent (1990):"Recursive Linear Models of Dynamic Economics," mimeo, Hoover Institution.

Judd, K. (1985): "Closed Loop Equilibrium in a Multi-Stage Innovation Race," Discussion Paper 647, Kellog Graduate School of Management, Northwestern University.

Judd, K. (1990(a)): "Cournot vs. Bertrand: A Dynamic Resolution," mimeo, Hoover Institution.

Judd, K. (1990(b)): "Minimum Weighted Residual Models for Solving Dynamic Economic Models," mimeo, Hoover Institution.

Lancaster, P. and K. Salkauskas (1986): *Curve and Surface Fitting*. San Diego: Academic Press.

Maskin E. and J. Tirole (1987): "A Theory of Dynamic Oligopoly, III: Cournot Competition," *European Economic Review,* 31, 947-968.

Maskin E. and J. Tirole (1988 (b)): "A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs," *Econometrica,* 56, ??-570.

Spence, A.M. (1977): "Entry, Capacity, Investment and Oligopolistic Pricing," *Bell Journal of Economics,* 8, 534-544.

Sulem, A. (1986):"Explicit Solution of a Two Dimensional Deterministic Inventory Problem," *Mathematics of Operations Research,* 11, 134-146.

Tirole, J. (1988): *The Theory of Industrial Organization.* Cambridge: MIT Press.

Villas-Boas, J.M. (1990): "Dynamic Duopolies with Non-Convex Adjustment costs," mimeo, Massachusetts Institute of Technology.y

SERIE DOCUMENTOS DE TRABAJO

The following working papers from recent years are still available upon request from:

Rocío Contreras,
Centro de Documentación, Centro De Estudios Económicos, El Colegio de México A.C., Camino al Ajusco # 20 C.P. 01000 México, D.F.

90/I Ize, Alain. "Trade liberalization, stabilization, and growth: some notes on the mexican experience".

90/II Sandoval Musi, Alfredo. "Construction of new monetary aggregates: the case of Mexico".

90/III Fernández, Oscar. "Algunas notas sobre los modelos de Kalecki del ciclo económico".

90/IV Sobarzo, Horacio E. "A consolidated social accounting matrix for input-output analysis".

90/V Urzúa, Carlos M. "El déficit del sector público y la política fiscal en México, 1980 - 1989".

90/VI Romero, José. "Desarrollos recientes en la teoría económica de la unión aduanera".

90/VII García Rocha, Adalberto. "Note on mexican economic development and income distribution".

90/VIII García Rocha, Adalberto. "Distributive effects of financial policies in Mexico".

90/IX Mercado, Alfonso and Taeko Taniura "The mexican automotive export growth: favorable factors, obstacles and policy requirements".

91/I Urzúa, Carlos M. "Resuelve: a Gauss program to solve applied equilibrium and disequilibrium models".

91/II Sobarzo, Horacio E. "A general equilibrium analysis of the gains from trade for the mexican economy of a North American free trade agreement".

91/III Young, Leslie and José Romero. "A dynamic dual model of the North American free trade agreement".

91/IV Yúnez-Naude, Antonio. "Hacia un tratado de libre comercio norteamericano; efectos en los sectores agropecuarios y alimenticios de México".

91/V Esquivel, Hernández Gerardo. "Comercio intraindustrial México-Estados Unidos".

91/VI Márquez, Colín Graciela. "Concentración y estrategias de crecimiento industrial".

92/I Twomey, J. Michael. "Macroeconomic effects of trade liberalization in Canada and Mexico".

92/II Twomey, J. Michael. "Multinational corporations in North America: Free trade intersections".

92/III Izaguirre Navarro, Felipe A. "Un estudio empírico sobre solvencia del sector público: El caso de México".

92/IV Gollás, Manuel y Oscar Fernández. "El subempleo sectorial en México".

92/V Calderón Madrid, Angel. "The dynamics of real exchange rate and financial assets of privately financed current account deficits"

92/VI Esquivel Hernández, Gerardo. "Política comercial bajo competencia imperfecta: Ejercicio de simulación para la industria cervecera mexicana".

93/I Fernández, Jorge. "Debt and incentives in a dynamic context".

93/II Fernández, Jorge. "Voluntary debt reduction under asymmetric information".

93/III Castañeda, Alejandro. "Capital accumulation games".