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Matching Through Institutions

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Abstract

We model a matching market with institutions – inspired by the assignment of social housing in Paris – as a three-sided market. Institutions own objects and have agents attached to them. Agents have preferences over objects. Objects have priorities over institutions. We show that fair assignments satisfying distributional constraints may fail to exist, and propose a sufficient condition – the over-demand condition – under which we prove existence. Existence derives from the construction of a new algorithm, the Nested Deferred Acceptance (NDA) algorithm, which combines a one-to-one matching between agents and objects and a one-to-many matching between objects and institutions. If interrupters are eliminated from the preference list, as in Kesten (2010), the NDA algorithm produces an assignment which is fair, Pareto optimal among fair assignments and strategy-proof for agents.

KEYWORDS: MATCHING, INSTITUTIONS, DEFERRED ACCEPTANCE ALGORITHM, SOCIAL HOUSING

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1 Introduction

This paper studies matching markets intermediated by institutions. Institutions own objects and agents are attached to institutions. In autarky, institutions assign their objects to their agents. But they may also consider a more flexible assignment by exchanging objects with other institutions, thereby increasing the choice set of all agents. In this paper, our objective is to study these flexible assignment rules, letting agents have preferences over the entire set of objects, objects having priorities over institutions, and institutions having preferences over the assignment of objects to agents. We also suppose that institutions face distributional constraints: the number of agents attached to one institution who receive an assignment must be equal to a fixed quota, typically equal to the number of objects that the institution initially owns.

Matching through institutions occurs in a large variety of situations. In Paris, social housing units are assigned according to a complex mechanism, where four different institutions pool their apartments and propose lists of applicants. These four institutions (the State, the city of Paris, local councils and private firms) contribute at different degrees to the financing of social housing and receive fixed quotas on the vacant apartments which are assigned every month to applicants.[1] The assignment of seats in study-abroad programs in American colleges and universities is another example of matching through institutions. Many liberal arts colleges maintain study-abroad programs with permanent staff and office space. Because all colleges cannot be present in all countries, they pool resources, allowing students from other colleges and universities to participate in their programs. These agreements are sometimes based on transfer payments and sometimes made on a quid pro quo basis, where the number of incoming and outgoing students are matched for each university.[2]

Increasing the choice of students is also the objective of inter-district school choice programs, like the one sponsored by the state of New Jersey.[3] These programs allow school districts to exchange students and to specialize in specific programs, like arts programs or programs for students with special needs. Pupils from one school district can be assigned to a school in another district up

[1]For a detailed description of the mechanism, see the annual report on assignment of social housing in Paris (APUR, 2014).

[2]In most cases, the agreements among colleges are bilateral. In some cases, like in the University of California system, there exists a central clearing house assigning students to the programs offered by different universities.

[3]See www.nj.gov/education/choice for a description of the system.

to a limit in the number of seats. Inter-district school programs are either based on financial transfers across school districts or, as in the case in New Jersey, on state funding. A final example of matching through institutions are tuition exchange programs for children of staff and faculty members of universities.[4] Through tuition exchange programs, children of faculty and staff can obtain a reduction in tuition when studying in other universities. The number of incoming and outgoing students in each university must be equalized, in order to maintain a financial balance among universities. A key difference between these applications is the role of institutions in the final assignment: they do not intervene in study abroad programs (see Abraham, Irving and Manlove (2007) for a model capturing the situation), they do in the Paris social housing programs where institutions keep track of households attached to them (formalized by priorities of institutions over household-apartment pairs) and their apartments (by setting priorities of institutions over their apartments, thus they can decide to keep top priority over their apartments, or limit the access to some institutions).

In this paper, we model matching through institutions as a three-sided market involving agents (which we refer to as "households"), objects (which we refer to as "apartments") and institutions. Our main objective is to obtain assignments which satisfy the classical properties of individual rationality, non-wastefulness, fairness and strategy-proofness. The main contributions of the paper are: 1. to allow priorities of institutions on household-apartment pairs to be perfect complements; 2. to propose a new algorithm, the Nested Deferred Acceptance (NDA) algorithm which extends deferred acceptance to matching through institutions. The NDA combines two different deferred acceptance algorithms. In the first deferred acceptance algorithm (the "outer loop"), each household asks for her most preferred apartment among those which have not yet rejected her. Given this list of demands, we run a second deferred acceptance algorithm (the "inner loop") among institutions. Each institution chooses a set of apartments that maximizes its preference and does not exceed its vector of quotas. If more than one institution is interested in one apartment, ties are broken according to the priority of each apartment. The process is repeated until apartments are assigned to households in such a way that institutions respect their quotas. Going back to the outer loop, we next ask rejected households to apply for their next preferred apartment and the procedure continues until no household is rejected.

We first prove that the NDA mechanism is strategy-proof for households and outputs a fair

[4] See Dur and Ünver (2014) for a description of these programs

assignment which is Pareto undominated by any fair assignment, when there is no distributional constraint. When we require to full fill a quota constraint, the distributional constraints may lead to nonexistence of feasible assignments. Even when feasible assignments exist, they may be very constrained, and can be subject to justified envy. In order to prove existence of fair assignments, we need to relax the distributional constraints by assuming that each institution has a sufficiently long list of agents so that every apartment is acceptable to some agent in the list. Under this sufficient condition – termed the over-demand condition – we show that fair assignments satisfying distributional constraints exist by extending the deferred acceptance algorithm to our three-sided market. We observe that, even under the over-demand condition, the NDA does not necessarily output a fair outcome. As in Kesten (2010), the problem comes from the presence of "interrupters" – institutions which are temporarily assigned to apartments that they will not be assigned to in the final matching, thereby blocking access of apartments to households from other institutions. Unlike Kesten (2010) our interrupters have to be deleted so as to reach a fair assignment, not Pareto efficiency, which means that it does not make sense to ask for consent of agents in our model.

The NDA with Interrupters (NDAI) improves the NDA by dropping apartments from the preferences of interrupter institutions. Our main result shows that the output given by the NDAI is fair, Pareto undominated by any other fair assignment satisfying the distributional constraints and strategy-proof. (Recall that, by contrast, the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) of Kesten is manipulable because interrupters are students who strategically report their preferences while in our case interrupters are institutions which do not submit preferences.)

We also analyze the case where the same agent can appear on the list of multiple institutions. (This is the case for example in the assignment of social housing in Paris). We show that fair assignments do not necessarily exist and that the NDAI then produces a matching which is not fair, but only fair among agents belonging to the same institution.

Even though the two models are not identical, our matching market bears strong similarities to the matching with distributional constraints studied in Kamada and Kojima (2015a), (2015b) and (2017). Inspired by the assignment of doctors to hospitals with flexible regional quotas, they propose a very general model where matching can be flexible across hospitals in the same region, and regions have preferences over the distribution of doctors to hospitals in their jurisdictions.

We show that the NDAI algorithm can be adapted to matching with distributional constraints, by reinterpreting regions as "institutions", doctors as "households" and jobs in hospitals as "apartments".

The matching with contracts approach (Hatfield and Milgrom (2005)) has been successful in relaxing the substitute condition by Kelso and Crawford (1982) and establishing that the Generalized Gale-Shapley algorithm outputs a stable allocation, even in settings where the set of stable matchings has no lattice structure, like in Hatfield and Kojima (2008 and 2010), Sönmez and Switzer (2013), Aygün and Sönmez (2012), Kominers and Sönmez (2016) and Hatfield and Kominers (2017). Contracts, however, are two-sided, which limits the scope of the analysis to pairwise stability (see example in Appendix A), which is not the natural stability restriction in our application. Our problem can also be modelled as a network, as in Ostrovsky (2008) and Hatfield and Kominers (2012), who extends the analysis and fixed point techniques to supply chain networks and establishes the stability of upstream-downstream contracts under same-side substitutability and a new substitutable condition, called cross-side complementary. Our three-sided approach is less general than Ostrovsky (2008) since we do not deal with networks of any size, and our preference/priority structure is also restricted. In contrast, it is less restrictive in term of complementarity. Rephrasing our problem in his setting, specifically section V.C, two-sided markets with complementarities, one side of the market can be interpreted as institutions that see both types of agents on the other side, households and apartments, as complements. While Ostrovsky (2008) and Hatfield and Kominers (2012) require households and apartments to be substitutable for one another, we dispense of this assumption since we deal with cases where an institution can allocate apartments exclusively to large families, for instance. Another difference lies in that fact that households have preferences defined on apartments, not on institutions. Matching with couples is another branch of the literature considering preferences/priorities exhibiting complementarities. Typically stability is not met but under restrictions on preferences/priorities (Blum, Roth and Rothblum (1998), Cantala (2004), Klaus and Klijn (2005), Pycia (2012), Sethuraman et al. (2006) and Nguyen and Vohra (2014)), or with high probability in large markets (Kojima, Pathak and Roth (2013), Ashlagi et al. (2014)).

The problem we consider is related to the rapidly emerging literature on matchings with constraints. In the school choice context, a number of recent papers have analyzed the effect of constraints resulting from affirmative action considerations. One stream of papers interprets affirmative action as "leveling the playing field", as in Kojima (2012) and Hafalir, Yenmez and Yildirim (2013) and Kominers and Sönmez (2013). Another stream of papers closer to our motivation consider affirmative action is an objective per se, formalized either by the existence of quotas as in Abdulkadiroğlu (2003), Abdulkadiroğlu, Sönmez (2003) and Hafalir, Yenmez and Yildirim (2013), or bounds as in Ehlers (2010), Ehlers, Hafalir, Yenmez and Yildirim (2014), Fragiadakis and Troyan (2015) and Bó (2016). One of the objective of these papers is to refine stability concepts and the deferred acceptance algorithms to conform to the bounds and quotas. Echenique and Yenmez (2015), Erdil and Kumano (2014), and Biró, Klijn and Pápai (2016) consider diversity as an objective of the school district and explore the tension between diversity objectives, stability and efficiency of priority systems and matching rules. Nguyen and Vohra (2017) have a different approach and use Scarf's Lemma to implement proportional distributional constraints.

Our analysis also bears a close connection to exchange markets with balanced constraints recently studied by Dur and Ünver (2016) and Biró, Klijn and Pápai (2015). These papers model exchange programs (like the tuition exchange for children of faculty members or the Erasmus exchange program' in European universities) where a balance must be kept between the number of incoming and outgoing students. Both papers consider a two-sided (rather than three-sided) matching problem, where colleges have preferences over students, and students have preferences over colleges. In Dur and Ünver (2016)'s tuition exchange model, students are ranked inside each college according to an exogenous priority (for example, the length of tenure of a faculty member). Dur and Ünver (2016) show that balancedness may lead to impossibility results, when associated with different natural axioms, like individual rationality or fairness. Their analysis focuses on efficiency and they propose a new procedure based on the Top Trading Cycle algorithm (rather than the Deferred Acceptance algorithm). Biró, Klijn and Pápai (2015) also focus attention on an extension of the TTC algorithm to analyze student exchange programs where a balancedness condition holds.

The paper is also related to the literature of three-sided markets. Unlike Alkan (1988) and Huang (2007), we assume that only the priorities of institutions are defined over the other two-sides of the market. To establish our existence result under quota restriction, unlike Biró and Mc Dermind (2010), we impose that all households only belong to one institution.

Finally, we note that the assignment of social housing that motivated our study has recently been analyzed in a series of papers (Leshno (2014), Bloch and Cantala (2016), Schummer (2016) and Thakral (2015)) which focus on very different aspects of the problem – the revelation of persistent information on types in Leshno (2014), the dynamic sequence of decisions in Bloch and Cantala (2016), the manipulation of orders in Schummer (2015) and multiple waiting list mechanisms in Thakral (2015).

As far as we are aware, we are the first one to deal with a pattern of perfect complements in matching markets and study the nesting of two Deferred Acceptance algorithm, which ties the acceptance of agents on the three sides of the market, While we deal with strict quotas in the main text, we show that it also applies to settings with maximum quotas, as in Kamada and Kojima (2015b). The NDAI, however, does not adapt to general distributional constraints, like Goto, Kojima, Tamura and Yokoo (2016).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents preliminary results on the existence of fair assignments and introduces the over-demand condition. Section 4 describes the Nested DA algorithm, and Section 5 contains our main results. In Section 6, we draw a comparison between our model of matching through institutions and matching with distributional constraints and regional preferences as in Kamada and Kojima (2015b). Section 7 contains our concluding comments. All proofs are collected in the Appendix.

2 The Model

A matching market with institutions is an 8−tuple $(I, Q, H, \tau, A, P, \succ, \pi)$ where:

1. $I = \{1, 2, ..., N\}$ is a finite set of institutions, a generic institution is i;

2. $Q = (q_i)_{i=1}^N$ is a vector of quotas, where q_i is the quota of institution i, a generic quota is q;

3. $H = \{h_1, ..., h_H\}$ is the finite set of households, a generic household is h;

4. $\tau : H \to I$ is a type function, which assigns to every household an institution $\tau(h)$. Conversely, $H(i) = \{h \in H \mid i = \tau(h)\}$ is the set of agents attached to institution i.[5]

[5] We assume in most of the analysis that $\tau(i)$ is a function– every agent is attached to a single institution. We consider the general case where $\tau(i)$ is a correspondence – households can be attached to multiple institutions in subsection 5.3.

5. $A = \{a_1, ..., a_A\}$ is the finite set of apartments, a generic apartment is a;

6. $P = (P_{h_1}, ..., P_{h_H})$ is a vector of households' preferences, P_h is the strict preferences of household $h \in H$ over $A \cup \{\varnothing\}$; $aP_h a'$ means that households h prefers a to a', an apartment a is acceptable for household h if $aP_h\varnothing$. Let $P_h : a_{1_h}, a_{2_h}, \ldots a_{A_h}$, and R_h be the antisymmetric preference list where $aR_h b$ and $bR_h a$ if and only if $a = b$;

7. $\pi = (\pi_a)_{a \in A}$ is the profile of priorities of apartments over institutions; π_a is the priority of apartment a over institutions $i \in I$. Let $A(i) = \{a | i\pi_a i' \text{ for all } i' \in I\}$ be the set of apartments where institution i has top priority.

8. For all $i \in I$, $\succ^i$ is the preference of institution i over sets of pairs (a, h) for $h \in H(i)$. Each institution has preferences over the assignment of apartments to its households. We write $(a, h) \succ^i \varnothing$ if the pair (a, h) is acceptable for institution i. This general formulation allows for different interpretations. Institutions may be benevolent, and inherit the preferences of the agents, so that $\succ^i$ is obtained directly from the preferences P_h of $h \in H(i)$. Alternatively, institutions may have their own fixed set of preferences, for example prioritizing among agents, or aiming at matching apartments to households with fixed characteristics independent of the preferences P_h, like the size of the household.

   We assume that for all $i \in I$ the priority $\succ^i$ is responsive on elements in $2^{A \times H_i}$, i.e. for all subsets of pairs $U \in 2^{A \times H_i}$ and all pairs (a_r, h_r), $(a_s, h_s) \in (A \times H_i) \setminus U$ we have that

   i. $U \cup (a_r, h_r) \succsim^i U \cup (a_s, h_s)$ if and only if $(a_r, h_r) \succsim^i (a_s, h_s)$, and

   ii. $U \cup (a_r, h_r) \succ^i U$ if and only if $(a_r, h_r) \succ^i \varnothing$.

When matching markets are intermediated by institutions, matchings are not simply defined as two-sided matchings between households and apartments, but as three-sided matchings associating households, institutions and apartments. The mechanism we propose consists in two separate assignments: we first assign apartments to institutions, then assign households to pairs consisting of one apartment and its matched institution. We thus describe an assignment as: (i) a many-to-one matching between apartments and institutions, and (ii) a one-to-one matching between households and pairs composed by one apartment and one institution. These assignments are formalized in the following definition.

An **assignment** $\mu = (\theta, \varphi)$ is a pair such that:

i. $\theta : A \cup I \to 2^A \cup I \cup \{\varnothing\}$ where

  i.a $\theta(a) \in I \cup \{\varnothing\}$,

  i.b $\theta(i) \in 2^A$ and $|\theta(i)| = q_i$,

  i.c $a \in \theta(i)$ if and only if $\theta(a) = i$;

ii. $\varphi : (A \times I) \cup H \to (A \times I) \cup H \cup \{\varnothing\}$, where

  ii.a $\varphi(h) \in A \times I \cup \{\varnothing\}$,

  ii.b $\varphi(a, i) \in H \cup \{\varnothing\}$,

  ii.c $\varphi(h) = (a, i) \Leftrightarrow \varphi(a, i) = h$. The corresponding projections are $\varphi_A(h) = a$ and $\varphi_I(h) = i$;

iii. $\theta(a) = i$ if and only if $\varphi(h) = (a, i)$ for some $h \in H$.

Conditions i. a, b and c define the many-to-one matching θ between apartments and institutions, taking into account the distributional constraint imposed by the quotas per institution. Conditions ii. a, b and c define the one-to-one matching φ between households and pairs composed by one apartment and one institution. Condition 3 defines a consistency condition between the two matchings, by requiring that whenever a household is assigned to a pair consisting of an apartment and an institution in φ, the apartment and institution are assigned to each other in θ.

The match of $h \in H$ is $\varphi(h) \in (A \times I) \cup \{\varnothing\}$, h is unmatched if $\varphi(h) = \varnothing$. The assignment of i is $\mu(i) = \{(a, h) \in A \times H : a \in \theta(i) \text{ and } \varphi_A(h) = a\}$.

We illustrate the definition of the assignments θ and φ with the following example. Consider $(I, Q, H, \tau, A, P, \succ, \pi)$ a market with institutions where $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, a vector of preferences $\succ^i$, a vector of preferences P and a profile of priorities π. The vector of quotas is $Q = (1, 2, 1)$, and the type function is given by $H(1) = \{h_3\}, H(2) =$

$\{h_4, h_5\}, H(3) = \{h_1, h_2\}$. A typical assignment for this market is represented as follows

$$\mu = \left(\begin{array}{ccccccc} h_3 & h_5 & h_1 & \varnothing & h_2 & h_4 & \varnothing \\ \overbrace{a_2}^{\varphi(h_3)} & \overbrace{a_4}^{\varphi(h_5)} & \overbrace{a_3}^{\varphi(h_1)} & \underbrace{\varnothing}_{\varphi^{-1}(\varnothing)} & \overbrace{\varnothing}^{\varphi(h_2)} & \overbrace{\varnothing}^{\varphi(h_4)} & \overbrace{a_1}^{\varphi(\varnothing)} \\ \underbrace{2 \quad\; 2}_{\theta(2)} & & \underbrace{3}_{\theta(3)} & \underbrace{1}_{\theta(1)} & \underbrace{\varnothing \quad \varnothing \quad \varnothing}_{\theta^{-1}(\varnothing)} & & \end{array} \right).$$

In this assignment, institution 1 ends up with no apartment, and hence does not fulfill its quota. Institution 2 is assigned the two apartments a_2 and a_4 which are given to households h_3 and h_5, and the quota is fulfilled. Institution 3 also fills its quota, obtaining apartment a_3 which is assigned to household h_1. Apartment a_1 remains unassigned.

We now define the choice functions of institutions. Consider sets of the form $\mathcal{U} = \{U \in 2^{A \times H(i)} | \text{neither households nor apartments are paired twice in } U\}$. The set $\mathcal{U}$ collects pairs of apartments and households attached to institution i such that every apartment and household only appear in one of the pairs. For any institution $i \in I$, we define the **choice function** Ch_i as a mapping choosing the pairs with the highest preference for i in $\mathcal{U}$: for all $(U, q_i) \in 2^{A \times H(i)} \times \mathbb{Z}_+$, the choice of i is the set $Ch_i(U, q_i) = \max_{\succ^i}\{u \subseteq U \mid (|u| = q_i) \text{ and } u \in \mathcal{U}\}$.

We now extend classical properties of the assignment μ to matching with institutions. An assignment μ is **individually rational** if

i. for all $h \in H$ either $\varphi_A(h) P_h \varnothing$ or $\varphi(h) = \varnothing$, and

ii. $\mu(i) = Ch_i(\mu(i), q_1)$.[6]

In words, an assignment is individually rational if both households and institutions – the two sides of the market which are endowed with preferences, prefer the outcome of matching μ to what they would get by not participating in the matching, $\varnothing$. We assume that the set of individually rational assignments is nonempty.

An assignment μ is **non-wasteful** if no household-institution pair (h, i) can claim an empty apartment a, i.e. there is no i, h and a such that:

i. $a P_h \varphi_A(h)$,

[6]Since priorities are responsive, this means that, for all $i \in I$, either $(a, h) \succ^i \varnothing$ for all $(a, h) \in \mu(i)$, or $\mu(i) = \varnothing$.

ii. $(a, h) \in Ch_i(\mu(i) \cup (a, h), q_i)$, and

iii. $\theta(a) = \varnothing$.

A household-institution pair (h, i) has **justified envy** over the household-institution pair (h', i') at the individually rational assignment μ if $i = \tau(h)$, $i' = \tau(h')$ and there exists $\varphi_A(h') = a \in \theta(i')$, such that

i. $aP_h\varphi_A(h)$,

ii. $(a, h) \in Ch_i(\mu(i) \cup (a, h), q_i)$, and

iii. $i\pi_a i'$.

We thus define deviations from pairs (h, i), where (h, i) claim an apartment a that improves preferences of both household h and institution i and for which institution i has higher priority than institution i' which is currently assigned to apartment a. Notice that we require that the new assignment obtained after the deviation satisfies the quota for the deviating institution. But the assignment following the deviation may not satisfy the distributional constraints, when the apartment a is initially assigned to an institution i' different form i. If we consider only deviations which do not require a change in assignment of apartments to institutions – and hence allows us to compare assignments which both satisfy the distributional constraints – we obtain a weaker form of envy, called justified envy over households of the same type:

There is **justified envy over households of the same type** when a pair (h, i) has justified envy over a pair (h', i).

We now define fair and efficient assignments:

An assignment μ is **fair** if it is individually rational, non-wasteful and there is no justified envy. A matching μ is **fair over households of the same type** if it is individually rational, non-wasteful and there is no justified envy for households of the same type.

An assignment μ is **Pareto efficient** if there is no matching μ' such that all households prefer μ' to μ, with strict inequality for at least one household. An assignment μ' **Pareto dominates** another assignment μ if $\mu'(h)R_h\mu(h)$ for each $h \in H$, and $\mu'(h')P_{h'}\mu(h')$ for at least one $h' \in H$.

Finally we consider the incentives of households to reveal their true preferences: A **mechanism** Λ associates a profile of preference lists with an assignment μ. Let R_h be the true preference list of each household h. The set of all possible preference lists of household h is denoted by $\Re_h$. A profile of preference list is a vector $R' = (R'_{h_1}, R'_{h_2}, \dots, R'_{h_H}) \in \Re_{h_1} \times \Re_{h_2} \times \cdots \times \Re_{h_H} = \Re$. As usual, R_{-h} is the profile of all preference lists except R_h. We do not treat symmetrically the revelation of preferences of households and institutions. We suppose that preferences of institutions are known, as would be the case when institutions have preferences which depend on fixed characteristics of households and apartments.

A mechanism Λ is **strategy proof for households** if telling the truth is a dominant strategy for all households, i.e. $\Lambda[R_h, R_{-h}](h) R_h \Lambda[R'_h, R_{-h}](h)$ for all $R'_h \in \Re_h$ and $R_{-h} \in \Re_{-h}$.

3 The Nested Deferred Acceptance Mechanism

In this Section, we extend the Deferred Acceptance (DA) algorithm to our setting and introduce the Nested Deferred Acceptance (NDA), which produces an assignment $\mu = (\theta, \varphi)$ in the matching market through institutions. The NDA will be used, as in Gale and Shapley (1962), to prove existence of a fair matching in our model.

The idea behind the NDA is to compute simultaneously a many-to-one matching, θ, and a one-to-one matching, φ, by nesting two deferred acceptance algorithms. In the main DA iteration (the "outer loop"), each unassigned household asks for her most preferred apartment. Given these demands, we run another DA (the "inner loop") where each institution demands a pair of apartments and households; then, the procedure continuous iteratively.

Formally, the NDA proceeds as follows:

Initialization

Consider a market $(I, Q, H, \tau, A, P, \succ, \pi)$. The assignment is initialized to be the empty assignment, so $\mu^0(i) = \mu^0(a) = \mu^0(h) = \varnothing$, i.e. $\theta^0(i) = \theta^0(a) = \varphi^0(h) = \varnothing$ for all $i \in I$, $a \in A$, $h \in H$.

Let $A_h^t = A$ and $t := 1$.

A. Eliciting the demand of households (the outer loop)

All unassigned households h ask for their most preferred apartment in A_h^t, denoted by D_h^t, while matched households h' iterate their demand to their match, i.e. $D_{h'}^t = \{\varphi_A^{t-1}(h')\}$.

For all $i \in I$ and $a \in A$, we define the set of households of type i that demand apartment a as follows:

$$H^t_{a,i} = \{h \in H \mid D^t_h = \{a\} \text{ and } i = \tau(h)\}.$$

The set of pairs (a, h) that can be assigned to institution i is defined as

$$M^t_i = \{(a, h) \in A \times H \mid (a, h) \succ^i \varnothing \text{ and } h \in H^t_{a,i}\}.$$

B. Iteration over $\mathbf{M^t_i}$ to match the demands of institutions and apartments (the inner loop)

Let $\theta^s(i) = \varnothing$ for all $i \in I$, and $\tilde{M}^s_i = M^t_i$, $s := 1$.

B.1 All institutions i demand the set of pairs $Ch_i(\tilde{M}^s_i, q_i)$. So, the set of institutions that demand an apartment a is

$$I^s_a = \{i \in I \mid \text{ there exists } (a, h) \in Ch_i(\tilde{M}^s_i, q_i)\}.$$

B.2 For all apartments a such that $I^s_a \neq \varnothing$, apartment a is assigned to the institution with the highest priority under π_a, i.e. $a \in \theta^s(i)$ if and only if $i = \max_{\pi_a} I^s_a$.

For all institutions i, let $\tilde{M}^{s+1}_i := \tilde{M}^s_i \setminus \{(a, h) \in \tilde{M}^s_i \mid (a, h) \in Ch_i(\tilde{M}^s_i, q_i) \text{ and } a \notin \theta^s(i)\}$. That is to say, we delete from the set $\tilde{M}^s_i$ those pairs where the institution i is rejected.

If $|\theta^s(i)| = q_i$ for all institutions i, or $\tilde{M}^{s+1}_i = \varnothing$ for all institutions i for which $|\theta^s(i)| < q_i$, go to B.3; otherwise, let $s = s + 1$ and go to B.1.

B.3 Rename $\theta^t(i) := \theta^S(i)$, where S is the last iteration of B.1. Furthermore, each pair (a, i) is tentatively assigned to household h if and only if $a \in \theta^t(i)$ and $(a, h) \in Ch_i(\tilde{M}^S_i, q_i)$. That is to say $\varphi^t(h) = (a, i)$.

C. Iteration over $\mathbf{D^t_h}$

For all unassigned households h, let $A^{t+1}_h := A^t_h \diagdown \{\max_{P_h} A^t_h\}$. If each household has been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the output assignment. Otherwise, $t := t + 1$, go to A.

The output of the previous mechanism depends on the market $E = (H, A, P, I, \succ, \pi_A, Q)$. So, it is denoted by $\mu^{NDA}[E] = (\theta^{NDA}[E], \varphi^{NDA}[E])$, or simply $\mu^{NDA} = (\theta^{NDA}, \varphi^{NDA})$ whenever there is no confusion. We use $NDA[P]$ to denote the NDA algorithm under the preference profile P. Note that the NDA algorithm has a finite number of steps because each DA ends in finite time.

3.1 An example of NDA

The following example shows how the NDA algorithm works, and the importance of Phase B (the inner loop).

Example 3.1. Consider the market $I = \{1, 2\}$, $H = \{h_1, h_2, h_3\}$ and $A = \{a_1, a_2\}$. The vector of quotas is $Q = (1, 1)$, and the type function is defined by $H(1) = \{h_1\}$ and $H(2) = \{h_2, h_3\}$. The profiles of institutions priorities, households preferences and apartments priorities is

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_1, h_2) \\ (a_2, h_1) & (a_2, h_3) \\ & (a_2, h_2) \end{pmatrix}, \; P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_1 & a_2 \\ a_2 & a_2 & a_1 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

At step A.1, households announce their preferred apartment: households h_1 and h_2 announce a_1, and household a_3 announces h_2. The NDA then moves to step B1.1, which is the inner loop. At this step, each institution announces its set of maximal matchings: 1 announces (a_1, h_1) and 2 announces $\{(a_1, h_2), (a_2, h_3)\}$. Observe that apartment a_1 is demanded by both institutions, so the algorithm moves to step B2.2. As institution 1 has priority over 2 for apartment a_1, a_1 rejects 2 and the assignment at the end of step B1.2 is $(a_1, h_1), (a_2, h_3)$. The NDA moves back to the outer loop and household h_2 announces apartment a_2 at step A.2. The algorithm moves now to step B2.1 where institution 2 announces that it prefers (a_2, h_3) to (a_2, h_2), and hence assignment $(a_1, h_1), (a_2, h_3)$ is produced. This is the last step of the algorithm as household h_2 has already announced all the apartments.

Remark 1. In the absence of the inner loop, in previous example, household h_3 is rejected from 2 at the end of step A.1. Thus, assignment $(a_1, h_1), (a_2, h_2)$ is produced at the end of A.2. Consequently, in this assignment $(h_3, 2)$ has justified envy over $(h_2, 2)$. $\square$

3.2 No distributional constraints

In this section we analyze the NDA algorithm when there are no distributional constraints – quotas are ineffective (this situation arises when institutions' quotas are large enough i.e. $q_i > A$ for all $i \in I$).

The implications for the NDA mechanism are presented below. We first show that in a market without distributional constraints, the "inner loop" of the NDA algorithm (phase B) only requires one step.

Lemma 3.1. *Consider a matching market with institutions and no distributional constraints, i.e.* $q_i > \#A$ *for all* $i \in I$. *Then, phase B of the NDA algorithm is iterated only once.*

Even more, when quotas are ineffective, all apartments tentatively assigned, at some step during the algorithm, are also assigned in the final assignment.

Lemma 3.2. *Consider a matching market with institutions and no distributional constraints, i.e.* $q_i > \#A$ *for all* $i \in I$. *If an apartment is assigned at some step* t *by some institution, this apartment is assigned under the assignment* μ^{NDA}.

The following theorem shows all the desirable properties that the NDA algorithm satisfies when there are no distributional constraints.

Theorem 3.1. *Consider a matching market with institutions* $(I, Q, H, \tau, A, P, \succ, \pi)$ *where* $q_i > \#A$ *for all* $i \in I$.

1. *The* μ^{NDA} *assignment is individually rational, non-wasteful and there is no justified envy; namely, the assignment* μ^{NDA} *is fair.*
2. *There is no fair assignment that Pareto dominates* μ^{NDA}.
3. *The NDA mechanism is strategy-proof for households.*

Theorem 3.1 shows that in the absence of distributional constraints, the model of matching through institutions inherits the properties of classical school choice problems. Even though the description of assignments is more complex in matching through institutions than in classical school choice problems, the properties of the assignment rule does not differ significantly.

Remark 2. The NDA works as a two sided market without quotas when the offering side is composed by households, and the accepting side by institutions and apartments: an apartment a is tentatively matched to household h via institution i if h is the top tentative matching of a for i, and i is the top institution through which an offer is emitted. If, in the final matching, institutions do not claim a fixed quota of assignments, the NDA's inner loop finishes in one step (see Lemma 3.1). In other words, institutions presence does not fundamentally change the assignment mechanism.

4 Existence and fairness with distributional constraints

In this Section, we provide examples to show that assignments satisfying distributional constraints and fairness may fail to exist. Distributional constraints require institutions to fill their quotas independently of households preferences, Evidently, as the following example shows, this may lead to inconsistencies.

Example 4.1. Let $I = \{1, 2\}$, $A = \{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3, h_4\}$, where households type function is given by $H(1) = \{h_1, h_2, h_3\}$ and $H(2) = \{h_4\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions priorities and households preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_3, h_4) \\ (a_3, h_2) & \end{pmatrix}, \; P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} \\ a_1 & a_1 & a_1 & a_1 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & a_2 & a_2 & a_3 \end{pmatrix}.$$

Notice that there is no assignment of apartment a_2 which is acceptable by any institution. This means that apartment a_2 cannot be assigned in an individually rational assignment and quotas cannot be fulfilled. □

Even when individually rational assignments satisfying distributional constraints exist, they may not satisfy fairness due to the preferences of institutions. We illustrate this point in the following example.

Example 4.2. Consider $I = \{1, 2\}$, $H = \{h_1, h_2, h_3\}$ and $A = \{a_1, a_2, a_3\}$. The vector of quotas is $Q = (q^1, q^2) = (2, 1)$, and the type function is given by $H(1) = \{h_1, h_2\}$ and $H(2) = \{h_3\}$. The profiles of institutions priorities, households preference and apartments priorities are

$$\begin{pmatrix} \succ^1 & \succ^2 \\ (a_2,h_2) & (a_1,h_3) \\ (a_1,h_1) & \\ (a_1,h_2) & \\ (a_3,h_1) & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_1 & a_1 \\ a_2 & a_2 & a_3 \\ a_3 & a_3 & a_2 \end{pmatrix}, \qquad \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Note that (a_1, h_3) is the unique acceptable assignment of institution 2 and (a_2, h_1) is not acceptable for institution 1. Therefore, in this market there exists one and only one assignment that satisfies the distributional constraints, which is:

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_3 & a_2 & a_1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Although μ satisfies the distributional constraints, note that i) $a_1 P_{h_1} a_3$, ii) $(a_1, h_1) \in Ch_1(\mu(1) \cup (a_1, h_1), q_1)$ because $(a_1, h_1) \succ^1 (a_3, h_1)$, and iii) $1 \pi_{a_1} 2$. Hence, the pair $(1, h_1)$ has justified envy over the pair $(2, h_3)$ at the apartment a_1. Consequently, the assignment μ is not fair. □

The previous examples build on the fact that distributional constraints may conflict with the preferences of households and institutions. In Example 4.1, the fulfillment of quotas would imply to assign an unacceptable apartment. In Example 4.2, the unique assignment which satisfies distributional constraints is upset by a deviation which results in one of the two institutions not fulfilling its quota.

The main difficulty highlighted by Example 4.2 is that institution 2 only views one apartment as acceptable. Assuming that every apartment is acceptable to any household and institution would clearly overcome this difficulty. We will provide a weaker requirement which guarantees the existence of a fair assignments satisfying distributional constraints. Specifically, it is enough to assume that for each institution and each apartment there exist a household who is willing to accept the apartment, and such that the institution also accepts the assignment. This leads us to define an over-demand condition, under which fair assignments satisfying the distributional constraints will be shown to exist.

Assumption 1. For all institutions i and apartments a, there is an unassigned household h such that $(a, h) \succ^i \varnothing$, $a P_h \varnothing$ and $i \pi_a \varnothing$.

Finally, we note that, perhaps surprisingly, the set of fair assignments does not have a lattice structure in our model. Households may disagree on the best fair assignment satisfying distributional constraints, as shown in the following example:

Example 4.3. Consider a market such that $H = \{h_1, h_2\}$, $A = \{a_1, a_2\}$ and $I = \{1, 2\}$, $H(1) = \{h_1\}, H(2) = \{h_2\}$.. Institutions priorities, households preferences and apartments priorities are given by.

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_2, h_2) \\ (a_2, h_1) & (a_1, h_2) \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} \\ a_1 & a_1 \\ a_2 & a_2 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In this market, we construct the following fair assignments where the distributional constraints hold.

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ 1 & 2 \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} h_2 & h_1 \\ a_1 & a_2 \\ 2 & 1 \end{pmatrix}.$$

We note that household h_1 prefers μ' to μ, whereas household h_2 has the reverse preferences. □

5 Main Results

5.1 Interrupters

The presence of distributional constraints may prevent the NDA from producing fair outcomes. During Phase B of the NDA algorithm, institutions make offers following the preferences of households attached to them and they might temporarily fill their quotas. Later in the run of the NDA, better options might arise for some institutions, leading them to drop some apartments. In this case, institutions act as interrupters, as defined in Kesten (2010): they temporarily accept pairs of households and apartments which will be dropped in the final outcome, preventing the emergence of fair outcomes.

The following example shows that the NDA may produce an assignment which fails to satisfy fairness over households of the same type.

Example 5.1. (There is justified envy over households of the same type). Let $I = \{1, 2\}$, $A = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, \ldots, h_7\}$, where households type function is given by

$H(1) = \{h_1, h_2, h_5, h_6\}$ and $H(2) = \{h_3, h_4, h_7\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions priorities and households preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1,h_1) & (a_2,h_3) \\ (a_1,h_2) & (a_2,h_4) \\ (a_2,h_1) & (a_1,h_3) \\ (a_2,h_2) & (a_1,h_4) \\ (a_1,h_5) & (a_1,h_7) \\ (a_2,h_6) & (a_2,h_7) \\ (a_3,h_1) & \\ (a_3,h_6) & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} & P_{h_6} & P_{h_7} \\ a_1 & a_2 & a_1 & a_1 & a_2 & a_1 & a_1 \\ a_2 & a_1 & a_2 & a_2 & a_1 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \end{pmatrix}.$$

The profile of apartment priorities is $\pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Running the NDA algorithm, the elicited demand of households, at step 1, is described in Table 1. Also, Table 2 shows the institutions that demand each apartment, where the apartment a_1 is demanded by both institutions.

I	$H^1_{a_1,i}$	$H^1_{a_2,i}$	$H^1_{a_3,i}$	$M^1_i = \tilde{M}^1_i$
1	h_1, h_6	h_2, h_5	$\varnothing$	$(a_1,h_1), (a_2,h_2)$
2	h_3, h_4, h_7	$\varnothing$	$\varnothing$	$(a_1,h_3), (a_1,h_4), (a_1,h_7)$

Table 1: A. Elicited Demand of Households at step 1.

Note that $2\pi_{a_1}1$, i.e., Phase B stops in one step. Consequently, the tentative assignment produced at the end of step 1 is given in the last column of Table 3.

The algorithm continues to step 2, where Table 4 shows the demands of households at this step, and Table 5 summarizes the institutions that demand each apartment. Note that institution 2 has a higher priority than institution 1 under priorities π_{a_1} and π_{a_2}. Moreover, we have that $q_2 = 1$ and $(a_2, h_4) \succ^i (a, h)$ for all $(a, h) \in M^2_2$. So, the tentative assignment produced at the end of step 2 is μ^2, which is shown in the last column of Table 6.

A	I_a^1
a_1	$1,2$
a_2	2
a_3	$\varnothing$

Table 2: B. Institutions demand at step 1.

I	$Ch_i^1(\tilde{M}_i^1, q_i)$	μ^1
1	$(a_1, h_1), (a_2, h_2)$	(a_2, h_2)
2	(a_1, h_3)	(a_1, h_3)

Table 3: B. Iteration over the sets M_i^1 at step 1.

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$	$M_i^2 = \tilde{M}_i^1$
1	h_5	h_2, h_6, h_1	$\varnothing$	$(a_2, h_1), (a_2, h_2)$
2	h_3	h_4, h_7	$\varnothing$	$(a_1, h_3), (a_2, h_4), (a_2, h_7)$

Table 4: A. Elicited Demand of Households step 2.

A	I_a^2
a_1	$1,2$
a_2	$1,2$
a_3	$\varnothing$

Table 5: B. Elicited demand of institutions at step 2.

I	$Ch_i^2(\tilde{M}_i^1, q_i)$	μ^2
1	(a_2, h_1)	$\varnothing$
2	(a_2, h_4)	(a_2, h_4)

Table 6: B. Iteration over the sets M_i^2 at step 2.

It is important to note that the assignments $(h_3, a_1, 2)$ and $(h_2, a_2, 1)$ are disrupted at the end of step 2 because $2\pi_{a_1}1$ and $(a_2, h_4) \succ^2 (a_1, h_3)$. Households h_2 and h_3 were rejected from apartments a_2 and a_1, respectively.

The algorithm continues to step 3 because not all households have been rejected by all their acceptable apartments. For example households h_2 and h_3 have not been rejected from the apartments a_1 and a_2, respectively. The demand of households at step 3 is shown in Table 7.

I	$H^3_{a_1,i}$	$H^3_{a_2,i}$	$H^3_{a_3,i}$	$M_i^3 = \tilde{M}_i^1$
1	h_2	$\varnothing$	h_1, h_5, h_6	$(a_1, h_2), (a_3, h_1), (a_3, h_6)$
2	$\varnothing$	h_3, h_4	h_7	$(a_2, h_3), (a_2, h_4), (a_3, h_7)$

Table 7: A. Elicited Demand of Households step 3.

By Table 7, we note that each apartment is demanded by a different institution. So, the tentative assignment produced at the end of step 3 is shown in the last column of Table 8.

I	$Ch_i^3(\tilde{M}_i^1, q_i)$	μ^3
1	$(a_1, h_2), (a_3, h_1)$	$(a_1, h_2), (a_3, h_1)$
2	(a_2, h_3)	(a_2, h_3)

Table 8: B. Iteration over the sets M_i^3 at step 3.

At the end of step 3, we note that h_5, h_6 and h_7 have been rejected from all their acceptable apartments. However, h_4 is rejected by a_2, and her last acceptable apartment is a_3. Thus, the algorithm continues to step 4, where h_4 demands the apartment a_3, and households h_1, h_2, h_3 iterate their demand to their match. Consequently, institutions 1 and 2 demand the apartment a_3 due to $(a_3, h_1) \succ^1 \varnothing$ and $(a_3, h_4) \succ \varnothing$. However, we know that institution 1 has a higher priority than institution 2 under the priority π_{a_3}, which implies that household h_4 is rejected from the apartment a_3 because $\tau(4) = 2$.

Therefore, the NDA algorithm stops at the end of step 4, and produces the assignment

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

We now observe that this assignment does not satisfy fairness over households of the same type. Consider institution 1, households h_1 and h_2, we know that $a_1 P_{h_1} a_3$ and $(a_1, h_1) \succ^1 (a_1, h_2)$, where $\varphi_I^{NDA}(h_1) = \varphi_I^{NDA}(h_2) = 1$, $\varphi_A^{NDA}(h_2) = a_1$ and $\varphi_A^{NDA}(h_1) = a_3$. The pair $(h_1, 1)$ has justified envy over $(h_2, 1)$ at apartment a_1.

We also note that the sets $\{(a_1, h_5), (a_2, h_6), (a_3, h_3)\}$ and $\{(a_1, h_4), (a_2, h_4), (a_1, h_7), (a_2, h_7)\}$ are sets of unassigned pairs that are acceptable for institutions 1 and 2, respectively. □

Example 5.1 illustrates how an interruption works. Observe that institution 2 is tentatively assigned to a_1 at step 1. Thus, household h_1 and institution 1 are displaced from a_1 at the end of step 1. Similarly, household h_3 and institution 2 are displaced from apartment a_1 at step 2 (see Table 4.) However, apartment a_1 is assigned to institution 1 at step 3. But household h_1 can no longer demand this apartment because she was rejected from it at step 1. As a consequence, the choice function of institution 1 does not consider the pair (a_1, h_1), which implies the presence of justified envy between the pairs $(h_1, 1)$ and $(h_2, 1)$. Following Kesten (2010)'s terminology, we say that institution 2 is an interrupter for apartment a_1. We formally define an interrupter below.

Given a matching problem to which the NDA is applied, we say that an institution i is an **interrupter for** apartment a if there exists

1. steps t to $t+n$ such that $a \in \theta^{t'}(i)$ for all $t' \in \{t, t+1, \ldots, t+n\}$ but $a \notin \theta^{t'}(i)$ for all $t' > t+n$, and

2. an institution $j \neq i$ and a household h such that $(a, h) \in Ch_j(M_j^{t'}, q_j)$ but $(a, h) \notin \mu^{t'}(j)$ for some $t' \in \{t, t+1, \ldots, t+n\}$.

5.2 Nested Deferred Acceptance with Interrupters

Following the Efficiency Adjusted Deferred Acceptance Mechanism (Kesten, 2010), we modify the NDA introducing a second stage where we search for all interrupter institutions. Then we let these institutions delete from their preference the pairs containing the apartment where they cause the interruption. We define the following delete operation on priorities $\succ^i$.

Let $\Im$ be the set of all possible priorities $\succ^i$, for all $i \in I$. The **delete operation** over $\Im$ is the function $\backslash : \Im \times (A \times H) \to \Im$ such that $\backslash(\succ, a)$, or simply $\succ \backslash a$, is the preference that declares all pairs $(a, h) \succ^i \varnothing$ as unacceptable for i. In other words, the preference $\succ \backslash a$ drops all acceptable pairs (a, h) from $\succ^i$ and holds the original order in the preference $\succ^i$. Note that Kesten (2010) defines this operation over students' preferences (the equivalent in our model of households in the outer loop), because he identifies that students causes the loss of efficiency during the Deferred Acceptance algorithm. In our case, the delete operation targets institutions, which are involved in the inner loop of the algorithm, as they are the source of the interruption of the NDA algorithm.

We finally formally present the Nested Deferred Acceptance with Interrupters (NDAI). Each step of this mechanism has two stages: the NDA algorithm runs in the first stage, while the second stage deletes pairs from the priorities of interrupters. The NDAI proceeds as follows.

Initialization

Initialize the counter of iterations over interrupter institutions at $x := 0$.

Step 0. This step is divided in the following stages:

Stage 0.1 NDA Phase. Let $\succ^0 = (\succ^{i,0})_{i\in I} = (\succ^i)_{i\in I}$. Run the NDA algorithm using the profile of priorities and preferences $(\succ^0, P)$.

Stage 0.2 Deletion in Priorities If there is no interrupter, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution i, $\succ^{i,1} = \succ^{i,0} \setminus a$; $\succ^{j,1} = \succ^{j,0}$ if j is not an interrupter.

Step x. The stages are the following.

Stage x.1 NDA Phase. Run the NDA algorithm with the profile of priorities and preferences $(\succ^x, P)$.

Stage x.2 Deletion in Priorities. If there is no interrupter, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution i, $\succ^{ix+1} = \succ^{i,x} \setminus a$; $\succ^{j,(x+1)} = \succ^{j,x}$ if j is not an interrupter.

The output of the previous mechanism is denoted by μ^{NDAI}. The NDAI is solvable in finite time because each NDA phase is solvable in a finite number of steps, and there are at most $|I|$ interrupters. Even more, NDAI is different from Kestens' EADAM because households ask for the same apartments at each NDAI iteration. The following example shows how the NDAI works.

Example 5.2. We consider the same market as in the Example 5.1 and consider stage 0.1 of the NDAI algorithm. We observe that institution 2 causes an interruption over the pair (a_1, h_1). Consequently, we delete all the pairs (a_1, h) from the priority $\succ^2$ at Stage 0.2. We get

$$\succ^{1,1} = \succ^1 \text{ and } \succ^{2,1} = \begin{bmatrix} (a_2, h_3) \\ (a_2, h_4) \\ (a_2, h_7) \end{bmatrix}.$$

Consider stage 1.1. We run the NDA algorithm with priorities $\succ^1$. Step 1.1 of this NDA algorithm is summarized in Table 9. Given that $q_1 = 2$, and $1\pi_{a_1}2$, the tentative assignment is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^{i1}_{a_1}$	$H^{i1}_{a_2}$	$H^{i1}_{a_3}$
1	h_1, h_6	h_2, h_5	$\varnothing$
2	$h_3,\ h_4, h_7$	$\varnothing$	$\varnothing$

Table 9: A. Elicited Demand Of Households step 1, Stage 1.1

The NDA in Stage 1.1 moves to step 2. Phase A of the algorithm is illustrated in the Table 10. Following the institutions priorities and the fact that $2\pi_{a_3}1$, the tentative assignment produced at the end of step 2 is

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & \varnothing & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$
1	h_1, h_5	h_2, h_6	$\varnothing$
2	$\varnothing$	h_3, h_4, h_7	$\varnothing$

Table 10: A. Elicited Demand Of Households step 2

Now, the NDA algorithm of Stage 1.1 moves to step 3. We show the demands of households in Table 11. Since the institution 1 has a higher priority than institution 2 under π_{a_3}, we get the tentative assignment μ^3.

$$\mu^3 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & a_3 & \varnothing & \varnothing \\ 1 & \varnothing & 2 & \varnothing & 1 & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^3_{a_1,i}$	$H^3_{a_2,i}$	$H^3_{a_3,i}$
1	h_1, h_2	$\varnothing$	h_5, h_6
2	$\varnothing$	h_3	h_4, h_7

Table 11: A. Elicited Demand Of Households step 3

I	$H^4_{a_1,i}$	$H^4_{a_2,i}$	$H^4_{a_3,i}$
1	h_1	$\varnothing$	h_2, h_6
2	$\varnothing$	h_3	$\varnothing$

Table 12: A. Elicited Demand Of Households step 4

Note that household h_2 has no been rejected by the apartment a_3, her last acceptable apartment. The NDA at Stage 1.1 moves to step 4 where Table 12 shows Phase A.

We observe that all households have been accepted or rejected at the end of the step 4, and hence Stage 1.1 stops. There are no interrupters because no apartment is rejected by any institution. Therefore, the NDAI algorithm stops and produces the following assignment.

$$\mu^{NDAI} = \mu^4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_3 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

This assignment satisfies fairness over households of the same type. □

Remark 3. NDAI AND EADAM are different. It is important to recall that institutions interrupter do not work in the same way that students interrupters in the EADAM. Previous example illustrates this fact; although institution 2 removes all acceptable pairs from its preferences, household h_1 still makes an offer to a_1 at iteration 1.1, and finally she gets it. On the contrary, following the EADAM we observe that h_1 gives up to get apartment a_1 because this apartment is deleted from her preferences, and consequently, h_1 makes an offer to a_2, her second best apartment, during iteration 1.1.

The next Theorem shows that the matching produced by the NDAI algorithm satisfies desirable properties.

Theorem 5.1. *Consider a matching market with institutions* $E = (I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$.

1. *The* μ^{NDAI} *is individually rational, non-wasteful, respects distributional constraints and there is no justified envy; namely, the assignment* μ^{NDAI} *is fair.*

2. *There is no assignment which is fair that Pareto dominates* μ^{NDAI}.

3. *The NDAI is strategy-proof for households.*

Theorem 5.1 extends the classical properties of the deferred acceptance algorithm to the model of matching through institutions. The proof, given in the Appendix, is an adaptation to our model of classical proofs of fairness and strategy-proofness of the Deferred Acceptance algorithm. Theorem 5.1 provides a strong rationale for the use of this mechanism to assign agents to apartments when institutions face distributional constraints.

5.3 Multiple institutions

In this subsection we generalize our previous results by allowing households to be attached to multiple institutions, as is the case for applications to social housing in Paris. We now assume that the type assignment mapping τ is a correspondence rather than a function: $\tau : H \to 2^I$, i.e $\tau(h) \subseteq I$ and $H_i = \{h \in H \mid i \in \tau(h)\}$. We first note that fairness is too demanding and must be weakened when agents can belong to multiple institutions. The following example, inspired by Biró and McDermind (2010), illustrates why fairness may fail.

Example 5.3. (**No fair assignments with multiple types**). Consider $H = \{h_1, h_2\}$, $I = \{1, 2\}$ and $A = \{a_1, a_2\}$. Households preferences, institutions priorities and apartments priorities are

$$P = \begin{pmatrix} P_{h_1} & P_{h_2} \\ a_1 & a_1 \\ a_2 & \end{pmatrix}, \quad \succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (h_1, a_1) & (h_1, a_2) \\ (h_2, a_1) & \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 1 & 2 \\ 2 & \end{pmatrix}.$$

The type function is defined as $H(1) = \{h_1, h_2\} = H(2)$. Note that the unique assignment that satisfies distributional constraints is

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_2 & a_1 \\ 2 & 1 \end{pmatrix}.$$

Note that $a_1 P_{h_1} a_2$, $(h_1, a_1) \succ^1 (h_2, a_1)$ and $1 \pi_{a_1} 2$. In other words, $(h_1, 1)$ has justified envy over $(h_2, 2)$ at apartment a_1.

Notice that the over-demand condition is satisfied in Example 5.3. Hence the inexistence of assignment satisfying fairness and distributional constraints comes from another source, here the fact that household h_1 belongs to the list of the two institutions which have different preferences over the apartment matched to household h_1. In order to preclude this phenomenon, we do not allow for envy involving households attached to two different institutions and consider the weaker requirement of fairness over households of the same type. We can then adapt Theorem 5.1 to show that the NDAI still satisfies desirable properties.

Theorem 5.2. *Consider a matching market with institutions $E = (I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$ where each institution is over demanded and households can belong to many institutions.*

1. *The μ^{NDAI} is individually rational, non-wasteful, respects distributional constraints and there is no justified envy over households of the same type; namely, the assignment μ^{NDAI} is fair over households of the same type.*
2. *There is no fair over households of the same type assignment that Pareto dominates μ^{NDAI}.*
3. *The NDAI is strategy-proof for households.*

6 Distributional constraints and regional preferences

In this section we show how the Nested Deferred Acceptance algorithm can be applied to the markets with distributional constraints and regional preferences introduced by Kamada and Kojima (2015b). This model is inspired by the assignment of doctors to hospitals in Japan. Instead of considering a simple many-to-one matching between hospitals and residents, Kamada and Kojima (2015b) note that there exists some flexibility in the way hospitals fill their quotas of positions, and introduce "regional preferences" over the pairs of hospitals and doctors. Markets with distributional constraints and regional preferences share some common features with matching markets through institutions. Since doctors are interested in hospitals, hospitals about doctors and regions care about the number of doctors that each hospital can accept, we note that doctors, regions and hospitals play a similar role as households, institutions and apartments in a market with institutions.

There are two main differences between Kamada and Kojima (2015b) and our model of matching through institutions. First, hospitals have priorities over doctors in Kamada and Kojima (2015b) whereas objects have priorities over institutions in our model. Second, and more importantly, regional preferences in Kamada and Kojima (2015b) are defined over the distribution of doctors over hospitals as measured by a vector of capacities whereas we suppose that institutions care about the precise assignment of households to objects. Notwithstanding those differences, we show that the NDAI algorithm can usefully be applied to produce stable assignments in markets with distributional constraints and regional preferences.

We now define the markets with distributional constraints and regional preferences of Kamada and Kojima (2015b). A market with distributional constraints and regional preferences $\tilde{E} = (D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ is defined by:

1. $D = \{d_1, d_2, \ldots, d_D\}$ is a finite set of doctors, a generic doctor is denoted by d;

2. $H = \{h_1, h_2, \ldots, h_H\}$ is a finite set of hospitals, a generic hospital is denoted by h;

3. $Q = (q_{h_1}, q_{h_2}, \ldots, q_{h_H})$ is a vectors of quotas, where q_h is the quota of the hospital h, a generic quota is q;

4. $R = \{1, 2, \ldots, R\}$ is a finite set of regions, a generic region is r;

5. $\tau : H \to R$ is the region function, i.e. if a hospital h belongs to the region r, we write that $\tau(h) = r$. Let H_r be the set of hospitals in region r, note that $H_r \cap H_{r'} = \varnothing$ for region $r' \neq r$;

6. $P = (P_{d_1}, P_{d_2}, \ldots, P_{d_D})$ is the vector of doctors' preferences, P_d is the strict preference of household $h \in H$ over $H \cup \varnothing$; hP_dh' means that doctor d prefers h to h', a hospital h is acceptable for doctor d if $hP_h\varnothing$.

7. $\succ = (\succ_{h_1}, \succ_{h_2}, \ldots, \succ_{h_H})$ is the profile of hospitals priorities over the set of doctors D. We assume that for each $h \in H$ the preference $\succ_h$ is responsive on 2^D, i.e. for any $d, d' \in D$ and $S \in 2^D$ we have that

   i. $S \cup \{d\} \succ_h S \cup \{d'\}$ if and only if $d \succ_h d'$, and

   ii. $S \cup \{d\} \succ_h S$ if and only if $d \succ_h \varnothing$;

8. $\succsim_r$ is the regional preference of r over the set of vectors $W_r = \{w = (w_h)_{h \in H} | w_h \in \mathbb{Z}_+\}$, where w_h specifies the number of doctors allocated to each hospital h in region r;

9. There exists a vector of regional caps $\tilde{Q} = (q_r)_{r \in R}$, where q_r is a non-negative integer for each region r.

Kamada and Kojima (2015b) introduce quasi-choice rules which pick the preferred capacity vector given the regional cap. Given $\succsim_r$, a function $\tilde{Ch}_r : W_r \times \mathbb{Z}_+ \to W_r$ is an **associated quasi choice rule** if $\tilde{Ch}_r(W_r, q_r) \in argmax_{\succeq_r}\{w \in W_r | \#w \le q_r\}$ for any non-negative $w = (w_h)_{h \in H_r}$. They also require that the quasi choice rule $\tilde{Ch}_r$ be **consistent**, that is, $\tilde{Ch}_r(\omega) \le \omega' \le \omega \Rightarrow \tilde{Ch}_r(\omega') = \tilde{Ch}_r(\omega)$. In other words, if $\tilde{Ch}_r$ is still available when the capacity vector reduces to $\omega' \le \omega$, then the associated quasi-choice rule chooses $\tilde{Ch}_r(\omega')$. They also assume that the regional preferences $\succeq_r$ satisfy the following regularity conditions:

(1) $\omega' \succ_r \omega$ if $\omega_h > q_h \ge \omega'_h$ for some $h \in H_r$ and $\omega'_{h'} = \omega_{h'}$ for all $h' \neq h$. In words, no hospital wants more doctor than its real capacity. This implies that $[\tilde{Ch}_r(w)]_h \le q_h$ for each $h \in H_r$.

(2) $\omega' \succ_r \omega$ if $\sum_{h \in H_r} \omega_h > q_r \ge \sum_{h \in H_r} \omega'_h$. So, each region prefers the total number of doctors in the region to be at most its regional cap.

(3) If $\omega' \lneq \omega \le q_{H_r} := (q_h)_{h \in H_r}$ and $\sum_{h \in H_r} \omega_h \le q_r$, then $\omega \succ_r \omega'$. In other words, each region prefers to fill as many positions of hospitals in the region while the regional cap would not be violated.

Regional preferences $\succsim_r$ are said to be **substitutable** if there exists an associated quasi choice rule $\tilde{Ch}_r$ that satisfies $w \le w' \Rightarrow \tilde{Ch}_r(w) \ge \tilde{Ch}_r(w') \wedge w$.

Next, Kamada and Kojima (2015b) define stable matchings in markets with distributional constraints and regional preferences:

A **matching** μ is a function that satisfies

(i) $\mu(d) \in H \cup \{\varnothing\}$ for all $d \in D$,

(ii) $\mu(h) \subseteq D$ for all $h \in H$ and

(iii) for any $d \in D$ and $h \in H$, $\mu(d) = h$ if and only if $d \in \mu(h)$.

A matching is **feasible** if $\mu(r) \le q_r$ for all $r \in R$, where $\mu_r = \bigcup_{h \in H_r} \mu(h)$.

A matching μ is **stable** if it is feasible, individually rational, and if (d, h) is a blocking pair, then

(i) $|\mu(h)| = q_{r_h}$,

(ii) $d' \succ_h d$ for all doctors $d' \in \mu(h)$, and

(iii) either $\mu(d) \notin H_{r(h)}$ or $w \succsim_{r(h)} w'$,

where $w_{h'} = |\mu(h')|$ for all $h' \in H_{r(h)}$ and $w'_h = w_h + 1$, $w'_{\mu(d)} = w_{\mu(d)} - 1$ and $w'_{h'} = w_{h'}$ for all other $h' \in H_{r(h)}$.

We adapt the NDA algorithm to the model of Kamada and Kojima (2015b). The algorithm differs from the NDA algorithm of our baseline model in two respects: (i) unmatched doctors are selected sequentially rather than simultaneously to make offers and (ii) we replace the choice function of regions by the quasi-choice function of regions. Formally:

Initialization

Consider a market $(D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ with distributional constraints and regional preferences. The matching is initialized to be the empty matching, so $\mu^0(h) = \mu^0(d) = \mu^0(r) = \varnothing$, for all $d \in D$, $h \in H$ and $r \in R$.

For all doctors $d \in D$, let $H_d^t := H$, and $t = 1$. For each region r, fix a quasi-choice rule $\tilde{Ch}_r$.

A. Eliciting the demand of doctors

Arbitrarily pick one unassigned doctor d, who asks for the most preferred hospital in H_d^t, denoted by D_d^t while r is the region of D_d^t; moreover matched doctors d' in region r iterate their demand to their match, $D_{d'}^t = \{\mu^{t-1}(d)\}$.

For all hospitals $h \in H$ in region r, we define the set of doctors that demand hospital h in region r as follows:

$$D_{h,r}^t = \{d \in D | D_d^t = \{h\}, d \succ_h \varnothing \text{ and } \tau(h) = r\}.$$

The set of pairs (d, h) that can be assigned to region r is defined as

$$M_r^t = \{(d, h) \in D \times H | d \succ_h \varnothing \text{ and } d \in D_{h,r}^t\}.$$

The possible assignments are

$$\mathcal{P}_r^t = \left\{p = \{(d, h)\}_{\tau(h)=r} \mid (d, h) \in M_r^t \text{ and } d \text{ is not matched twice}\right\}.$$

The number of doctors matched to hospital h at p is $w_h(p) = \#\{d \in D \mid (d, h) \in p\}$, thus, the set of capacity vectors is

$$W_r^t = \left\{w = (w_h)_{h \in H} | \exists\ p \in \mathcal{P}_r^t \text{ and } w_h = w_h(p) \text{ for all } h \text{ in region } r\right\}.$$

B. Matching the demand of region r and hospitals of the region.

B.1 Regions r demands the vector

$$\omega_r^t = (\omega_h^t)_{\tau(h)=r} = \tilde{Ch}_r(W_r^t, q_r).$$

B.2 Each hospital h in region r is tentatively assigned to the preferred subset of $D_{h,r}^t$ with cardinality w_h^t. The assignment in other regions remains the same.

C. Iteration over $\mathbf{D_d^t}$

Let $H_d^{t+1} := H_d^t \setminus \{\max_{P_d} H_d^t\}$, $t := t + 1$.

If all doctors have been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the outcome assignment. Otherwise, go to the Phase A.

The assignment produced by the previous algorithm is denoted by $\tilde{\mu}^{NDA}$. It depends on a market $\tilde{E}$ and a fixed associated quasi choice rule $\tilde{Ch}$. We now state that the NDA produces a stable and strategy-proof matching in the market with distributional constraints and regional preferences:

Theorem 6.1. *Suppose that regional preferences $\succeq_r$ are substitutable for all $r \in R$. Then the matching produced by the nested deferred acceptance algorithm is stable and strategy proof for doctors.*

Proof. See Online Appendix. □

Remark 4. It is important to note that Kamada and Kojima establish a relation between distributional constraints and contracts. Thus, they derive their results using the Cumulative Offer Process which considers contracts between hospitals and doctors, while regions determine the number of doctors attached to each hospital. In the matching through institutions model, although it is possible to define contracts considering triads (a, h, i), apartments do not determine the number of households attached to each institution. Even more, since there are no property rights, contract (a, h, i) summarizes two contracts, one between apartments and institutions, and other between institutions and households. Consequently, when households, apartments and priorities take care of the others, and not only in the number of contract accepted, it is necessary to modify the COP to show the interaction within them.

7 Concluding Remarks

We model a matching market with institutions as a three-sided market. Institutions own objects and have agents attached to them. Agents have preferences over objects. Objects have priorities over institutions. We show that fair assignments satisfying distributional constraints may fail to exist, and propose a sufficient condition – the over-demand condition – under which we prove existence. Existence derives from the construction of a new algorithm, the the Nested Deferred Acceptance (NDA) algorithm, which combines a one-to-one matching between agents and objects and a one-to-many matching between objects and institutions. If interrupters are eliminated from the preference list, as in Kesten (2010), the NDA algorithm produces an assignment which is fair, Pareto optimal among fair assignments and strategy-proof for agents.

The model of matching through institutions we consider is inspired by the assignment of social housing in Paris but the procedure we propose applies more generally to situations where agents belong to different groups, and pool their resources to obtain more flexible outcomes. We hope to study more applications – like the exchange of students across universities or of pupils across school districts in detail in future work.

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A Pairwise stability

Definition 1. Contracts are **bilateral substitutes** for hospital h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X - Y$, such that $z \notin Ch_h(Y \cup \{z\})$ and $z \in Ch_h(Y \cup \{x, z\})$, then $z \in Y_D$ or $x_D \in Y_D$.

Definition 2. Contracts are **unilateral substitutes** for hospital h if for all $z \in X - Y$, $z \notin Ch_h(Y \cup \{z\})$ and $z \in Ch_h(Y \cup \{x, z\})$ then $z \in Y_D$.

Definition 3. Contracts satisfy the **irrelevance of rejected contracts** for hospital h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X - Y$, such that $z \notin Ch_h(Y \cup \{z\})$, then $Ch_h(Y) = Ch_h(Y \cup \{z\})$.

Definition 4. Contracts satisfy the **law of aggregate demand** for hospital h if, for all $X', X'' \subseteq X$, we have that

$$X' \subset X'' \to |Ch_h(X')| \leq |Ch_h(X'')|.$$

The following example, based on the Example 1 of (Hatfield and Kojima, 2010), satisfies previous conditions.

Example A.1. Consider a problem with $H = \{h_1, h_2, h_3\}$ and $D = \{d_1, d_2, d_3\}$. Doctors preferences over contracts are

$$P_D = \begin{pmatrix} P_{d_1} & P_{d_2} & P_{d_3} \\ (h_1, 1/2) & (h_2, 1/2) & (h_3, 1/2) \\ (h_2, 1/2) & (h_3, 1/2) & (h_1, 1/2) \end{pmatrix}.$$

Hospitals preferences over contracts are

$$\succ_H = \begin{pmatrix} \succ_{h_1} & \succ_{h_2} & \succ_{h_3} \\ (d_1, 1/2)(d_2, 1/2) & (d_1, 1/2)(d_2, 1/2) & (d_1, 1/2)(d_3, 1/2) \\ (d_1, 1) & (d_2, 1) & (d_3, 1) \\ (d_2, 1) & (d_3, 1) & (d_1, 1) \\ (d_3, 1/2) & (d_1, 1/2) & (d_2, 1/2) \end{pmatrix}.$$

Note that each hospital prefers two half-time doctors to one full-time doctor. Thus, the Cumulative Offer Process outputs an assignment that is blocked by the set $\{h_2, d_1, d_2\}$. In other words, d_2 prefers half-time in h_2 to half time in h_3, i.e. d_2 is better off with h_2. Also, hospital h_2 prefers two half-time contracts with d_1 and d_2, to a half-time contract with h_1. That is to say, this assignment is not core stable.

In general, following the previous reasoning, there is no core stable assignments in this example.

B Proofs

Proof of Lemma 3.1:

Consider a step t of the NDA algorithm. We know that phase B starts with the set of acceptable apartment-households pairs $\tilde{M}_i^1 = M_i^t$ for all institutions i. This implies that each institution i demands the set $Ch_i(\tilde{M}_i^t, q_i)$, a set where apartments and households are not paired twice.

Let $A_i^t = \{a \in A | (a,h) \in M_i^t\}$ be the set of apartments demanded by some household of type i at step t. Since $|A_i^t| \leq A$, there are no distributional constraints and priorities are responsive, for all apartments a all top acceptable pairs (a,h) belong to the set $Ch_i(\tilde{M}_i^1, q_i)$. Each institution i demands all the apartments in the set A_i^t. So, all the apartments in $\bigcup_{i \in I} A_i^t$ belong to $\theta^1(j)$, for some $j \in I$, at the end of Phase B.2.

Consequently

$$\tilde{M}_i^2 = \tilde{M}_i^1 \setminus \{(a,h) \in \tilde{M}_i^1 | a \in \theta^1(j) \text{ for some } j \neq i\} = \tilde{M}_i^1 \setminus \tilde{M}_i^1 = \varnothing$$

for all institution i. Therefore, phase B stops in one iteration.

Proof of Lemma 3.2: Consider an apartment a that is assigned by some institution i at some step t, i.e. a belongs to $\theta^t(i)$. Since households iterate their demand to their match, we have that $i \in I_a^{t+1}$. In other words, this apartment is demanded by some institution at step $t+1$. Since there are no distributional constraints and priorities are responsive, the apartment a is assigned to some institution at the end of step $t+1$ (the institution in I_a^{t+1} with the highest priority at π_a). Iterating this argument, we conclude that the apartment a is assigned to some institution at all steps $t' \geq t$.

Therefore $\mu^{NDA}(a) \neq \varnothing$ because the NDA algorithm stops in a finite number of steps.

Proof of Theorem 3.1:

Individual Rationality. For all institutions $i \in I$, we know that $\mu^{NDA}(i) \subseteq Ch_i(M_i^T, q_i)$, where T is the last iteration of the NDA algorithm. Thus, $(a,h) \succ^i \varnothing$ for all $(a,h) \in \mu^{NDA}(i)$. Therefore, $\mu^{NDA}(i) \succ \varnothing$ for all $i \in I$.

Moreover, the NDA algorithm stops when every unmatched household has been rejected by all her acceptable apartments, in this case $\varphi(h) = \varnothing$, or every household is matched to some acceptable apartment, i.e. $\varphi_A(h) P_h \varnothing$ for all $h \in H$.

Non-wastefulness. We proceed by contradiction. We assume the existence of a household-institution pair (h,i) that claims an empty apartment a. That is to say, we have that i) $a P_h \varphi_A(h)$, ii) $(a,h) \in Ch_i(\mu^{NDA}(i) \cup (a,h), q_i)$, and iii) $\theta^{NDA}(a) = \varnothing$.

The condition i) means that household h demands the apartment a at some step t of the NDA algorithm. Moreover, condition ii) guarantees that the pair (a,h) is acceptable for the institution i. Thus, institution i demands the apartment a at step t. Applying Lemma 3.2, there exists some institution j such that $a \in \theta^{NDA}(j)$ which contradicts the condition iii).

There is no justified envy. Suppose, on the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i'), where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment a such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i')$, and

i. $aP_h\varphi_A^{NDA}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup (a, h), q_i)$,

iii. $i\pi_a i'$.

By condition i), household h demands the apartment a at some step t. Moreover, condition ii) ensures that the pair (a, h) is acceptable for the institution i, i.e. $(a, h) \in M_i^t$. Consequently, we have that $i \in I_a^t$. We analyze the following cases.

Case 1. $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi_A^{NDA}(h') = a$, household h' demands the apartment a at some step t'. Even more, $(a, h') \in Ch_i(M_i^{t'}, q_i)$ because (a, h') belongs to the set $\mu^{NDA}(i)$. Given that $(a, h) \in M_i^t$ but $\varphi_A^{NDA}(h) \neq a$, the responsiveness of priorities ensure that

$$(a, h') \succ^i (a, h). \tag{1}$$

Moreover, since no apartment can be paired twice, condition ii) implies that

$$(a, h) \succ^i (a, h'),$$

in contradiction with (1).

Case 2. $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$. We know that $a \in \theta^{NDA}(i')$, which implies the existence of some step t' where

$$i'\pi_a j \text{ for all } j \in I_a^t, \text{ for all } t \geq t',$$

according to Phase B.3. In particular

$$i'\pi_a i,$$

in contradiction with condition iii).

In all cases we get a contradiction, therefore there is no justified envy at the assignment μ^{NDA}. So, this assignment is fair.

Pareto undominated. We proceed as in Gale and Shapley (1962). To prove that μ^{NDA} is Pareto undominated, we show that in any other fair assignment, each household gets the same apartment or an apartment less preferred than $\varphi^{NDA}(h)$.

An apartment a is said to be **achievable** for a household h if there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu_A(h) = a$. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm.

Hypothesis of induction. At step t we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

Induction step. Consider that some household h^* is rejected at step $t+1$ from an apartment a. We assume, on the contrary, that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair (a, h^*) is acceptable for the institution i^*.

Now, let h be the household assigned to the apartment a at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a, i)$ where $i = \tau(h)$. We analyze the following cases.

Case 1. $i = i^*$. Since $\varphi^{t+1}(h) = (a, i)$, the apartment a belongs to $\theta^{t+1}(i)$. That is to say

$$(a, h) \succ^i (a, h^*) \tag{2}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that

$$(a, h) \in Ch_i(\mu(i) \cup (a, h)).$$

Note that h prefers a to all the apartments that have not rejected her, then the induction hypothesis ensures that household h prefers a to any other achievable apartment for her:

$$aP_h\varphi^\mu_A(h).$$

Moreover, $(a, h^*) \in \mu(i)$. That is to say, the pair (h, i) has justified envy over the pair (h^*, i) at the apartment a in the assignment μ, which contradicts the fact that μ is a fair assignment.

Case 2. $i \neq i^*$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I^{t+1}_a$. Moreover, $i \in I^{t+1}_a$ because $\varphi^{t+1}(h) = (a, i)$. Given that $a \in \theta^{t+1}(i)$, we conclude that $a \notin \theta^{t+1}(i^*)$ because $i = \max_{\pi_a} I^{t+1}_a$; thus

$$i\pi_a i^*. \tag{3}$$

By the induction hypothesis, we know that household h strictly prefers a to any other achievable apartment for her, i.e.

$$aP_h\varphi^\mu_A(h). \tag{4}$$

Now, we know that $(a, h) \in \mu^{t+1}(i)$, this means that the pair (a, h) is acceptable for institution i. Moreover, $(a, h^*) \in \theta^{\mu}(i^*)$, thus a is not assigned to i at μ. Also, there are no distributional constraints and institutions priorities are responsive, then

$$(a, h) \in Ch_i(\mu(i) \cup (a, h), q_i). \tag{5}$$

By 3, 4 and 5, the pair (h, i) has justified envy over the pair (h^*, i^*) at the apartment a in the assignment μ, which contradicts the fact that μ is fair.

In any case, a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. Therefore, μ^{NDA} is Pareto undominated by fair assignments.

Truth-telling is a dominant strategy for households. We construct the proof as in Roth (1982).

For each household h, we say that P'_h is a **successful** misrepresentation of P_h if P'_h is a preference list such that

$$\varphi_A^{NDA}[P'_h, P_{-h}](h) P_h \varphi_A[P](h).$$

Let $a' := \varphi_A^{NDA}[P'_h, P_{-h}](h)$, we define the preference list P''_h where the apartment a' is declared as the most preferred apartment of h. Let P' and P'' be the preference profiles where household h reports P'_h and P''_h, respectively, and other households do not change their true preferences. The following lemma establishes that P'_h and P''_h are equivalents in the sense that

$$\varphi_A^{NDA}[P'_h, P_{-h}](h) = \varphi_A^{NDA}[P''_h, P_{-h}](h).$$

Lemma B.1. *Consider a matching market through institutions with no distributional constraints. Then* $\varphi_A^{NDA}[P'_h, P_{-h}](h) = \varphi_A^{NDA}[P''_h, P_{-h}](h)$.

Proof. By paragraphs above, the assignment $\mu^{NDA}[P']$ is fair with respect to P'. Let $i := \tau(h)$, it is no possible at $\varphi_A^{NDA}[P'_h, P_h]$ that (h, i) has justified envy over other pairs under P'' because no apartment is preferred to a' under P''_h. Since other preferences did not change between P' and P'', it implies 1. $\mu^{NDA}[P']$ is fair with respect to the profile P'', i.e. a' is achievable for h under the preference profile P''; 2. the apartment a' is the best acceptable achievable apartment of h under P'', and since $\mu^{NDA}[P'']$ is Pareto undominated by other fair assignments, we conclude that

$$\mu^{NDA}[P'](h) = \mu^{NDA}[P''](h).$$

□

The following Lemma establishes that households are not worse off when a household successfully misrepresents her true preference list.

Lemma B.2. *Consider P'_h a preference list different from the true preference list of h. If $\varphi_A^{NDA}[P'](h)$ is weakly preferred to $\varphi_A^{NDA}[P](h)$, then for each household $h' \neq h$, either*

$$\varphi_A^{NDA}[P'](h') P_{h'} \varphi_A^{NDA}[P](h') \text{ or } \varphi_A^{NDA}[P'](h') = \varphi_A^{NDA}[P](h').$$

Proof. We proceed by contradiction, i.e. we assume that $aP_{h'}a'$ for some $h' \in H$, where

$$\varphi_A^{NDA}[P'](h') = a' \text{ and } \varphi_A^{NDA}[P](h') = a \text{ for } h' \neq h.$$

So, there exists a step t of $NDA[P']$ at which h' is rejected from a. Hence, by Lemma 3.2, the apartment a is assigned to some household h''. We analyze the following cases.

Case 1. If $\tau(h'') = \tau(h') = i$. Since $\mu^{NDA}[P']$ is fair, we have that $(a, h'') \succ^i (a, h')$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we have the following sub-cases.

Case 1.1 $aP_{h''}a''$. Then (h'', i) has justified envy over (h, i) at apartment a in the assignment $\mu^{NDA}[P]$, a contradiction.

Case 1.2 $a''P_{h''}a$. By Lemma 3.2, there exists a household h''' such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on h'' to h''', which will end up generating either a contradiction, or an infinite succession of households $\{h^{(k)}\}$. This is not possible because H is finite.

Case 2. If $i = \tau(h') \neq \tau(h'') = i'$. Since $\mu^{NDA}[P']$ is fair, we have that $i'\pi_a i$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we have the following sub-cases.

Case 2.1 $aP_{h''}a''$. Then (h'', i') has justified envy over (h, i) at apartment a in the assignment $\mu^{NDA}[P]$, a contradiction.

Case 2.2 $a''P_{h''}a$. By Lemma 3.2, there exists a household h''' such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on h'' to h''', which will end up generating either a contradiction, or an infinite succession of households $\{h^{(k)}\}$. This is not possible because H is finite.

Therefore, no household h' is worse off under the assignment $\mu^{NDA}[P']$. □

We are ready to prove that truth-telling is a dominant strategy for all households. We assume, on the contrary, the existence of household h^* and a successful misrepresentation P'_{h^*} of P_{h^*}. That is to say

$$a'P_{h^*}a,$$

where $\varphi_A^{NDA}[P](h^*) = a$ and $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}] = a'$.

By Lemma B.1, we consider that P_{h^*} is the preference list where a' is the most preferred apartment. Our objective is to show that P'_{h^*} is not a successful manipulation. To do that, we follow the proof of Roth (1982) about the strategy proofness of the DA. We say that household h **makes a match** at step t of the NDA algorithm, if h demands $\varphi_A^{NDA}(h)$ at step t. This proof analizes the possible steps where h^* makes a match.

Lemma B.3. *Consider the household h^* with preferences P_{h^*} and P'_{h^*} such that*

$$\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi_A^{NDA}[P](h^*).$$

If $\varphi_A^{NDA}[P](h^0) = \varnothing$ then $\varphi_A^{NDA}[P'](h^0) = \varnothing$.

Proof. By contradiction, suppose that h^0 gets an apartment at P'. Since assignments $\mu^{NDA}[P]$ and $\mu^{NDA}[P']$ are non-wasteful, this means that some household h, previously matched at P, is unmatched at P', violating Lemma B.2. □

First, if a household makes a match in the last step of the NDA algorithm, under the true preferences, then no manipulation is a successful misrepresentation of her true preference list.

Claim B.1. Suppose that h^* makes a match at t^*, with $1 \leq t^* \leq T$, then $\varphi_A^{NDA}[P'](h) = \varphi_A^{NDA}[P](h)$ for all h that makes a match at T. Moreover, if h^* makes a match at T, there is no profitable deviation P'_{h*} of her true preference list P_{h^*}.

Proof. First, we present the argument for h. Let T be the last step of the $NDA[P]$ and consider that the household h makes a match at step T, say $a = \varphi_A^{NDA}[P](h)$. Since $\mu^{NDA}[P]$ is non-wasteful, all apartments are matched, at $T-1$ either

1. a is unmatched, or

2. a is matched to a household h_1 who is unmatched at $\mu^{NDA}[P]$.

Case 1. Since apartment a was unmatched at $T-1$, all matched households prefer their match at $\mu^{NDA}[P]$ to a. By Lemma B.2, this implies that none of them gets a at $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$. By Lemma B.3 all unmatched households are still unmatched. So, by non-wastefulness, h gets a and does not strictly improve her match under the profile (P'_{h^*}, P_{-h^*}).

Case 2. Let h_1 be the household matched with a at $T-1$, who is unmatched at T. For all matched $h_2 \neq h_1$, if any, who prefer a to $\varphi_A^{NDA}[P](h_2)$, we have that $(a, h_1) \succsim_{\tau(h_1)} (a, h_2)$, by fairness of $\mu^{NDA}[P]$, or $\tau(h_1)\pi_a\tau(h_2)$. Thus, if h strictly improves under (P'_{h^*}, P_{-h^*}), then h_1, or an unmatched household, gets a or an apartment preferred to a at $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$ because the assignment is fair; in contradiction to Lemma B.3.

In both cases we conclude that $\varphi_A^{NDA}[P'_{h^*}, P_{h^*}](h) = \varphi_A[P](h)$.

Second, the argument is the same for h^* that makes a match at T since other households h do not improve her allocation under (P'_{h^*}, P_{-h^*}) (Lemma A.2). □

Now, we consider that h^* makes a match at some step t of the $NDA[P]$ procedure, with $1 \leq t < T$. We show that no household, matched after t, changes its final allocation when household h^* misrepresents her true preference list through P'_{h^*}.

Claim B.2. Suppose that h^* makes a match at t^*, $1 \leq t^* \leq T$ and P'_{h^*} and P_{h^*} are such that

$$\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi_A^{NDA}[P](h^*).$$

Then $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^t) = \varphi_A^{NDA}[P](h^t)$ for $h^t \neq h^*$ who makes a match at t, where $t^* \leq t \leq T$.

Proof. The proof is by induction.

Base of Induction. Starts in $t = T$. It is true by Claim B.1.

Hypothesis of Induction. Suppose the property is true until step $t+1$.

Induction Step. Let a^t be the match of h^t at $\varphi_A^{NDA}[P](h^t)$.

Case 1. a^t is unmatched at $t-1$. Since a^t is unmatched at $t-1$, all households matched before/at t strictly prefer their match to a^t, by Lemma B.2 they do not get a^t at P'. By induction hypothesis and Lemma B.3, those who make a match after t get the same apartment or nothing, i.e. non-wastefulness of μ^{NDA} guarantees that $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^{t-1}) = \varphi_A^{NDA}[P](h^{t-1})$.

Case 2. a^t is matched at $t-1$. Let h^{t-1} be the match of a^t at $t-1$; thus h^{t-1} has top priority among households who prefer a^t to their match and make a match before/at t. By Lemma B.3, fairness of $\varphi_A^{NDA}[P']$ and Lemma B.2, h^{t-1} or a household h' that makes a match after t should get a^t, or an apartment prefer to a^t, at $\varphi_A^{NDA}[P']$. Since h^{t-1} and h' makes a match, if any, after t, it is not the case by the hypothesis of induction. □

In any step where household h^* makes a match, Claim B.2 implies that $\varphi_A^{NDA}[P'](h^*) = \varphi_A^{NDA}[P](h^*)$. That is to say, there is no successful misrepresentation of P_{h^*}, and the NDA is strategy-proof.

Proof of Theorem 5.1 : Let x^* be the last iteration of the NDAI mechanism, i.e. there are no interrupter at the end of the NDA phase $x^*.1$. The last step of the NDA phase is denoted by T.

Individual Rationality. For all institutions $i \in I$, we know that $\mu^{NDAI}(i) \subseteq Ch_i(M_i^T, q_i)$, where T is the last iteration of the NDA phase. Thus, $(a, h) \succ^i \varnothing$ for all $(a, h) \in \mu^{NDAI}$. Therefore, $\mu^{NDAI}(i) \succ \varnothing$ for all $i \in I$. Moreover, the NDA algorithm stops when every unmatched household has been rejected by all her acceptable apartments, in this case $\varphi(h) = \varnothing$, or every household is matched to some acceptable apartment, i.e. $\varphi_A(h) P_h \varnothing$ for all $h \in H$.

Distributional Constraints. Consider an institution i that does not fulfill its quota in the assignment μ^{NDAI}. Thus, $|\mu^{NDAI}(i)| < q_i$, i.e. there is at least one apartment a that remains unassigned. We know that μ^{NDAI} is IR, and the over-demand condition holds in the market E, then for all institutions i and, particularly, the apartment a, there exists an unassigned household h such that (a, h) is acceptable for i and $a P_h \varnothing$. Considering the NDA phase at Stage $x^*.1$ we have that (a, h) belongs to M_i^t, for some step t, or not.

Case 1. If $(a, h) \in M_i^t$, since i did not fulfill its quota and (a, h) is acceptable for i whose priorities are responsive, it means that (a, h) has not been assigned to i; this implies the existence of an institution j such that $j \pi_a i$, i.e. $a \in \theta(j)^{t'}$ for some step $t' \geq t$. However, we know that $a \notin \mu^{NDAI}(j)$. Hence, the institution j is an interrupter for a, which contradicts the fact that there are no interrupters at x^*.

Case 2. $(a, h) \notin M_i^t$ because of $\succ^{i, x^*}$, i.e. the priority of institution i at stage x^* since the NDAI algorithm actualizes institutions priorities. This case only happens if i is an interrupter over a and fulfill its quota, thus (a, h) is deleted from $\succ^{i,x}$ at some stage x of the NDAI algorithm. This is not possible because the institutions did not fulfill its quota.

In any case we get a contradiction. Therefore, we have that $|\mu^{NDAI}(i)| = q_i$ for all $i \in I$.

Non-wastefulness. Follows from the fact that each institution fills its quota.

There is no justified envy. Let x^* be the last iteration of the NDAI. Suppose, on the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i'), where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment a such that $\varphi_A^{NDAI}(h') = a$, $a \in \theta(i')$, and

i. $a P_h \varphi_A^{NDAI}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup (a, h), q_i)$,

iii. $i \pi_a i'$.

By condition i), household h demands the apartment a at some step t. Moreover, the condition ii) ensures that the pair (a, h) is acceptable for the institution i, i.e. $(a, h) \in M_i^t$. Consequently, we have that $i \in I_a^t$. We analyze the following cases.

Case 1. $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi_A^{NDAI}(h') = a$, household h' demands the apartment a at some step t'. Moreover condition 1 ensures that h also demands the apartment a at some step t', by responsiveness of preferences at each step the institution i picks the top pair with a, thus

$$(a, h') \succ^i (a, h). \tag{6}$$

Moreover, since no apartment can be paired twice and the fact that priorities are responsive, the condition ii) implies that $(a, h) \succ^i (a, h')$ in contradiction with (6).

The following Lemma is required to prove Case 2.

Lemma B.4. *Consider an institution i with priorities $\succ^i$ and suppose that $\#\tau(h) = 1$ for all $h \in H$. If i is an interrupter over an apartment a through a household h, then (h, i) has not justified envy over other pairs (h^a, i^a) at μ^{NDAI}, where $(a, h^a) \in \mu^{NDAI}(i^a)$.*

Proof of Lemma B.4. Since i is an interrupter, there exist steps $\underline{t}$ and $\overline{t}$ such that $(a, h) \in \mu^t(i)$ for all $t \in [\underline{t}, \overline{t}]$, but $(a, h) \notin \mu^{t'}(i)$ for all $t' \geq \overline{t}$.

Because priorities of institutions are responsive, at $\overline{t} + 1$ institution i drops (h, a) only if it has filled its quota and $(h^{\overline{t}+1}, a^{\overline{t}+1}) \succ^i (h, a)$ for all $(h^{\overline{t}+1}, a^{\overline{t}+1}) \in \mu^{\overline{t}+1}(i)$.

Moreover, institutions only improve all along the sequence of tentative matchings, thus $(\overline{h}, \overline{a}) \succsim^i (\overline{h}^{\overline{t}+1}, a^{\overline{t}+1})$ for all $(\overline{h}, \overline{a}) \in \mu^{NDAI}(i)$ and $(\overline{h}^{\overline{t}+1}, a^{\overline{t}+1}) \in \mu^{\overline{t}+1}(i)$, and fills its quota at μ^{NDAI}, so $(h, a) \notin Ch_i(\mu^{NDAI}(i) \cup (h, a), q_i)$, thus (h, i) has not justified envy over (h^a, i^a) at μ^{NDAI}, where $(a, h^a) \in \mu^{NDAI}(i^a)$. $\square$

Case 2. $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$.

Case 2.1 Consider that i is not an interrupter, then i demands apartment a at iteration x^*. Also, we know that $a \in \theta^{NDAI}(i')$, then there exists a step t' such that $i'\pi_a j$ for all $j \in I_a^t$, for all $t \geq t'$, according to the Phase B.3. Since $a \notin \theta^{NDAI}(i)$ we have that $i'\pi_a i$, in contradiction with the condition iii).

Case 2.1 If i is an interrupter, assuming that (h, i) has justified envy over (h', i'), through apartment a, contradicts Lemma B.4.

In any case we get a contradiction, therefore there is no justified envy at the assignment μ^{NDA}. So, this assignment is fair.

Pareto Undominated for fair assignments. We proceed as in Gale and Shapley (1962). To prove that μ^{NDA} is Pareto undominated, we show that in any other fair assignment, each household gets the same apartment or an apartment less preferred than $\varphi^{NDA}(h)$.

An apartment a is said to be **achievable** for a household h if there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu_A(h) = a$. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm at iteration x^* where there are not interrupters.

Hypothesis of induction. At step t, we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

Induction step. Consider that some household h^* is rejected at step $t+1$ from an apartment a. We assume, on the contrary, that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair (a, h^*) is acceptable for the institution i^*.

Now, let h be the household assigned to the apartment a at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a, i)$ where $i = \tau(h)$. We analyze the following cases.

Case 1. $i = i^*$. Since $\varphi^{t+1}(h) = (a, i)$, the apartment a belongs to $\theta^{t+1}(i)$. That is to say

$$(a, h) \succ^i (a, h^*) \tag{7}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that $(a, h) \in Ch_i(\mu(i) \cup (a, h), q_i)$.

Note that h prefers a to all the apartments that have not rejected her, then the induction hypothesis ensures that household h prefers a to any other achievable apartment for her $aP_h\varphi^\mu_A(h)$. Moreover, $(a, h^*) \in \mu(i)$. That is to say, the pair (h, i) has justified envy over the pair (h^*, i) at the apartment a in the assignment μ, which contradicts the fact that μ is a fair assignment.

Case 2. $i \neq i^*$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I^{t+1}_a$. Since there are not interrupters at iteration x^*, the offer of i^* is rejected only if i fulfills its quota. So, there exist $(a_{\hat{x}}, h_{\hat{x}}) \in \mu^{t+1}(i^*)$ such that $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} (a, h^*)$ where $\hat{x} = 1, 2, \ldots, q_{i^*}$. We know that $(a, h^*) \in \mu(i^*)$, i.e., some pair $(a_{\hat{x}}, h_{\hat{x}}) \notin \mu(i^*)$. By hypothesis of induction, this implies that $h_{\hat{x}}$ prefers $a_{\hat{x}}$ to any other achievable apartment for her, i.e.

$$a_{\hat{x}} P_{h_{\hat{x}}} \varphi_A^{\mu}(h_{\hat{x}}). \tag{8}$$

Since $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} \varnothing$, by expression (8), we conclude that the pair $(h_{\hat{x}}, i^*)$ has justified envy over the pair (h^*, i^*) in the assignment μ, which contradicts the fact that μ is fair.

In any case, a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. Therefore, μ^{NDA} is Pareto undominated by fair assignments.

Strategy Proofness. For each household h, we say that P'_h is a **successful** misrepresentation of P_h if P'_h is a preference list such that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) P_h \varphi_A^{NDAI}[P](h)$. Let $a' := \varphi_A^{NDAI}[P'_h, P_{-h}](h)$, we define the preference list P''_h where the apartment a' is declared as the most preferred apartment of h. Let P' and P'' be the preference profiles where household h reports P'_h and P''_h, respectively, and other households do not change their true preferences.

Let P'_h be a misrepresentation of P_h where apartment a is acceptable. We analyze the following cases.

Case 1. The institution i is an interrupter for apartment a at some iteration x. As a consequence, all pairs (a, h) are deleted from the preference $\succ^{i,x}$, i.e., pairs (a, h) are not acceptable at preference $\succ^{i,x'}$ for all $x' > x$. So, household h is not assigned to apartment a in stages $x'.1$, with $x' > x$. Therefore, P'_h is not a successful misrepresentation of P_h.

Case 2. The institution i is not an interrupter for apartment a. To prove that P'_h is not a successful misrepresentation, first, we prove the following Lemma.

Lemma B.5. *Consider a matching market where the over-demand condition holds. If an apartment is assigned at some step t by some institution, this apartment is assigned under the assignment μ^{NDAI}.*

Proof of Lemma B.5: Suppose that x^* is the last iteration of the NDAI where there are no interrupters. Consider an apartment a that is assigned by some institution i at some step t, during the NDA phase of stage x^*, i.e. a belongs to $\theta^t(i)$. Since households iterate their demand to their match, we have that $i \in I_a^{t+1}$. In other words, this apartment is demanded by some institution at step $t+1$. Since the over-demand condition holds, there are no interrupters and priorities are responsive, the apartment a is assigned to some institution at the end of step $t+1$

(the institution in I_a^{t+1} with the highest priority at π_a). Iterating this argument, we conclude that the apartment a is assigned to some institution at all steps $t' \geq t$. $\square$

Therefore $\mu^{NDAI}(a) \neq \varnothing$ because the NDA algorithm stops in a finite number of steps. We construct the proof as in Roth (1982). The following lemma establishes that P'_h and P''_h are equivalents in the sense that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = \varphi_A^{NDAI}[P''_h, P_{-h}](h)$.

Lemma B.6. *Consider a matching market through institutions where the over-demand condition holds. Then* $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = \varphi_A^{NDAI}[P''_h, P_{-h}](h)$.

Proof. By paragraphs above, the assignment $\mu^{NDAI}[P']$ is fair with respect to P'. Let $i := \tau(h)$, it is no possible at $\varphi_A^{NDAI}[P'_h, P_h]$ that (h, i) has justified envy over other pairs under P'' because no apartment is preferred to a' under P''_h. Since other preferences did not change between P' and P'', it implies that: 1. $\mu^{NDAI}[P']$ is fair with respect to the profile P'', i.e. a' is achievable for h under the preference profile P''; 2. the apartment a' is the best acceptable achievable apartment of h under P'', and since $\mu^{NDAI}[P'']$ is Pareto undominated by other fair assignments, we conclude that $\mu^{NDAI}[P'](h) = \mu^{NDAI}[P''](h)$. $\square$

The following Lemma establishes that households are not worse off when a household successfully misrepresents her true preference list.

We are ready to prove that truth-telling is a dominant strategy for all households. We assume, on the contrary, the existence of household h^* and a successful misrepresentation P'_{h^*} of P_{h^*}. That is to say $a' P_{h^*} a$, where $\varphi_A^{NDAI}[P](h^*) = a$ and $\varphi_A^{NDAI}[P'_{h^*}, P_{-h^*}] = a'$.

By Lemma B.6, we consider that P'_{h^*} is the preference list where a' is the most preferred apartment. Our objective is to show that P'_{h^*} is not a successful manipulation. To do that, we follow the proof of Roth (1982) about the strategy proofness of the DA. We say that household h **makes a match** at step t of the NDA algorithm, if h demands $\varphi_A^{NDAI}(h)$ at step t. This proof analizes the possible steps where h^* makes a match.

First, if a household makes a match in the last step of the NDA algorithm, under the true preferences, then no manipulation is a successful misrepresentation of her true preference list.

Claim B.3. Let x^* be the last iteration of the NDAI. Suppose that h^* makes a match at t^*, with $1 \leq t^* \leq T$, then $\varphi_A^{NDAI}[P'](h) = \varphi_A^{NDAI}[P](h)$ for all h that makes a match at T. Moreover, if h^* makes a match at T, there is no profitable deviation P'_{h*} of her true preference list P_{h^*}.

Proof. First, we present the argument for h. Let T be the last step of the $NDA[P]$ and consider that the household h makes a match at step T, say $a = \varphi_A^{NDAI}[P](h)$. Since $\mu^{NDAI}[P]$ is non-wasteful, all apartments are matched, at $T-1$ either

1. a is unmatched, or

2. a is matched to a household h_1 who is unmatched at $\mu^{NDAI}[P]$.

Case 1. Since apartment a was unmatched at $T-1$, all matched households prefer their match at $\mu^{NDAI}[P]$ to a. By Lemma B.2, this implies that none of them gets a at $\mu^{NDAI}[P'_{h^*}, P_{-h^*}]$. By Lemma B.3 all unmatched households are still unmatched. So, by non-wastefulness, h gets a and does not strictly improve her match under the profile (P'_{h^*}, P_{-h^*}).

Case 2. Let h_1 be the household matched with a at $T-1$, who is unmatched at T. For all matched $h_2 \neq h_1$, if any, who prefer a to $\varphi_A^{NDAI}[P](h_2)$, we have that $(a, h_1) \succsim_{\tau(h_1)} (a, h_2)$, by fairness of $\mu^{NDAI}[P]$, or $\tau(h_1)\pi_a\tau(h_2)$. Thus, if h strictly improves under (P'_{h^*}, P_{-h^*}), then h_1, or an unmatched household, gets a or an apartment preferred to a at $\mu^{NDAI}[P'_{h^*}, P_{-h^*}]$ because the assignment is fair; in contradiction to Lemma B.3.

In both cases we conclude that $\varphi_A^{NDAI}[P'_{h^*}, P_{h^*}](h) = \varphi_A[P](h)$.

Second, the argument is the same for h^* that makes a match at T since other households h do not improve her allocation under (P'_{h^*}, P_{-h^*}) (Lemma A.2). □

Now, we consider that h^* makes a match at some step t of the $NDA[P]$ procedure, with $1 \leq t < T$. In any step where household h^* makes a match, Claim B.2 implies that $\varphi_A^{NDAI}[P'](h^*) = \varphi_A^{NDAI}[P](h^*)$. That is to say, there is no successful misrepresentation of P_{h^*}, and the NDAI is strategy-proof.

Proof of Theorem 5.2:

Distributional Constraints. See the proof of Theorem 5.1.

Non-wastefulness. Follows from the fact that each institution fills its quota.

There is no justified envy over households of the same type. Let x^* be the last iteration of the NDAI. Suppose, on the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i), where $i \in \tau(h) \cap \tau(h')$. Then, there exists an apartment a such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i)$, and

i. $aP_h\varphi_A^{NDA}(h)$,

ii. $(a,h) \in Ch_i(\mu^{NDA}(i) \cup (a,h), q_i)$.

The proof is analogous to the proof of Case 1 in Theorem 5.1.

Note that Case 2 can not be extended when households are attached to multiple institutions because it a household h can form a blocking pair with another institution $i' \neq i$, which is what happens in example 5.3.

Pareto Undominated for fair over the same type assignments. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm at iteration x^* where there are not interrupters.

Hypothesis of induction. At step t, we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

Induction step. Consider that some household h^* is rejected at step $t+1$ from an apartment a. We assume, on the contrary, that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair (a, h^*) is acceptable for the institution i^*.

Now, let h be the household assigned to the apartment a at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a, i)$ where $i = \tau(h)$. We analyze the following cases.

Case 1. $i \in \tau(h^*)$. Since $\varphi^{t+1}(h) = (a, i)$, the apartment a belongs to $\theta^{t+1}(i)$. That is to say

$$(a, h) \succ^i (a, h^*) \tag{9}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that $(a,h) \in Ch_i(\mu(i) \cup (a,h), q_i)$. Note that h prefers a to all the apartments that have not rejected her, then the induction hypothesis ensures that household h prefers a to any other achievable apartment for her $aP_h\varphi^\mu_A(h)$. Moreover, $(a, h^*) \in \mu(i)$. That is to say, the pair (h, i) has justified envy over the pair (h^*, i) at the apartment a in the assignment μ, which contradicts the fact that μ is a fair assignment.

Case 2. $i \notin \tau(h^*)$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I_a^{t+1}$. Since there are not interrupters at iteration x^*, the offer of i^* is rejected only if i fulfills its quota. So, there exist $(a_{\hat{x}}, h_{\hat{x}}) \in \mu^{t+1}(i^*)$ such that $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} (a, h^*)$ where $\hat{x} = 1, 2, \ldots, q_{i^*}$. We know that $(a, h^*) \in \mu(i^*)$, i.e., some pair $(a_{\hat{x}}, h_{\hat{x}}) \notin \mu(i^*)$. By hypothesis of induction, this implies that $h_{\hat{x}}$ prefers $a_{\hat{x}}$ to any other achievable apartment for her, i.e.

$$a_{\hat{x}} P_{h_{\hat{x}}} \varphi_A^{\mu}(h_{\hat{x}}). \tag{10}$$

Since $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} \varnothing$, by expression (10), we conclude that the pair $(h_{\hat{x}}, i^*)$ has justified envy over the pair (h^*, i^*) in the assignment μ, which contradicts the fact that μ is fair.

In any case, a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. Therefore, μ^{NDA} is Pareto undominated by fair assignments.

Strategy-proofness. The prove is analogous to Theorem 5.1. In the following Lemma we generalize the Case 1 of strategy-proofness' proof in Theorem 5.1 when households are attached to multiple institutions.

Claim B.4. Consider a market through institutions with distributional constraints where the over-demand condition holds. Consider a pair (a, h) such that $aP_h\varphi_A^{NDAI}[P](h)$. There is no misrepresentation P'_h of P_h such that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = a$.

Proof. Let P'_h be a misrepresentation of P_h where apartment a is acceptable. We analize the following cases.

Case 1. All institutions in $i \in \tau(h)$ are interrupters for the apartment a. As a consequence, all pairs (a, h) are deleted from the preference $\succ^{i,x_i}$, i.e., pairs (a, h) are not acceptable at preference $\succ^{i,x'}$ for all $x' > x_i$. So, household h is not assigned to apartment a in stages $x'.1$, with $x' > x_i$. Therefore, P'_h is not a successful misrepresentation of P_h.

Case 2. There exists an institution $i \in \tau(h)$ such that i is not an interrupter for apartment a. We proceed as in Theorem 5.1 to prove that no misrepresentation of P_h is successful. □

Therefore, Claim B.4 implies that the NDAI algorithm is strategy-proof. □

Online Appendix to "Matching Through Institutions"

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1 Proof of Theorem 6.1.

We adapt the proof of Kamada and Kojima (2015b) to the Nested Deferred Acceptance algorithm with regional preferences.

First, as they do, we establish the relation between matching markets with regional preferences and matching with contracts. So, let $X = D \times H$ be the set of contracts. Note that, for each doctor d, the preference profile P_h induces a preference relation $\tilde{P}_h$ over $(\{d\} \times H) \cup \{\varnothing\}$ in the following way $(d, h')\tilde{P}_d(d, h)$ if and only if $h' P_d h$. Moreover, we say that $(d, h)\tilde{P}_d\varnothing$ if hospital H is unacceptable under P_d.

Now, for each region $r \in R$, we define preferences $\succeq_r$ and its associated choice rule $\overline{Ch}_r$ over all subsets of $D \times H_r$. For any $X' \subset D \times H_r$, let $\omega(X') := (w_h(X'))_{h \in H_r}$ be the vector such that $w_h(X') = |\{(d, h) \in X' | d \succ_h \varnothing\}|$. For each X', the chosen set of

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contracts $\overline{Ch}_r(X')$ is defined by

$$\overline{Ch}_r(X') = \bigcup_{h \in H_r} \left\{ (d,h) \in X' \middle| |\{d' \in D | (d',h) \in X', d' \succeq_h d\}| \leq (\tilde{Ch}_r(\omega(X')))_h \right\}. \tag{1}$$

That is, each hospital $h \in H_r$ chooses its $(\tilde{Ch}_r(\omega(X')))_h$ most preferred contracts available in X'. The domain of the choice rule $\overline{Ch}_r$ can be extended to all subsets of X by

$$\overline{Ch}_r(X') = \overline{Ch}_r(\{(d,h) \in X' | h \in H_r\})$$

for any $X' \subseteq X$.

Definition 1. (Hatfield and Milgrom (2005)). Choice rule $\overline{Ch}_r(\cdot)$ satisfies the **substitutes condition** if there does not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \overline{Ch}_r(X' \cup \{x'\})$ and $x' \in \overline{Ch}_r(X' \cup \{x, x'\})$.

Definition 2. (Hatfield and Milgrom (2005)). Choice rule $\overline{Chr}(\cdot)$ satisfies the **law of aggregate demand** if for all $x' \subseteq X'' \subseteq X$, $|\overline{Ch}_r(X')| \leq |\overline{Ch}_r(X'')|$.

Proposition 1.1. *Suppose that $\succeq_r$ satisfies the substitutes condition. Then the choice rule $\overline{Ch}_r(\cdot)$ defined above satisfies the substitutes condition and the law of aggregate demand.*

Proof. The Nested Deferred Acceptance algorithm is not related with the choice rule $\overline{Ch}_r$. So, this proposition is taken from Proposition 1 of Kamada and Kojima (2015b). □

A subset X' of $X = D \times H$ is said to be **individually rational** if (1) for any $d \in D$, $|\{(d,h) \in X' | h \in H\}| \leq 1$, and if $(d,h) \in X'$ then $hP_d\varnothing$, and (2) for any $r \in R$, $\overline{Ch}_r(X') = X' \cap (D \times H_r)$.

Definition 3. A set of contracts $X' \subseteq X$ is a **stable allocation** if

(1) it is individually rational, and

(2) there exists no region $r \in R$, hospital $h \in H_r$, and a doctor $d \in D$ such that $(d,h)\tilde{P}_d x$ and $(d,h) \in \overline{Ch}_r(X' \cup \{(d,h)\})$, where x is the contract that d receives at X' if any and $\varnothing$ otherwise.

When condition (2) is violated by some (d, h), we say that (d, h) is a **block** of X'. A doctor-optimal stable allocation in the matching model with contracts is a stable allocation that every doctor weakly prefers to every other stable allocation (Hatfield and Milgrom, 2005).

Given any individually rational set of contracts X', define a corresponding matching $\mu(X')$ in the original model by setting $\mu(d)(X')) = h$ if and only if $(d, h) \in X'$; and $\mu(d)(X') = \varnothing$ if and only if no contract associated with d is in X'. Since each doctor regards any set of contracts with cardinality of at least two as unacceptable, each doctor receives at most one contract at X' and hence $\mu(X')$ is well defined for any individually rational X'.

Proposition 1.2. *If X' is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

Proof. See Proposition 2 of Kamada and Kojima (2014b). □

Remark 1. It is is important to recall the connection between the Nested Deferred Acceptance and the Cumulative Offer Process (Hatfield and Milgrom, 2005). That is to say, if doctor d asks for her most preferred hospital h at some step in the NDA, then contract (d, h) is proposed at the same stpe of the cumulative offer process. Also, the set of doctors accepted by a hospital at some step of the NDA is equivalent to the set of contracts held at the corresponding step of the cumulative offer process. Thus, if X' is the output of the cumulative offer process, then $\mu(X')$ is the matching generated by the NDA.

Now, we are ready to continue with the proof of Theorem 6.1. By Proposition 1, the choice function of each region satisfies the substitutes condition and the law of aggregate demand in the associate model of matching with contracts. By Hatfield and Milgrom (2005), Hatfield and Kojima (2009), and Hatfield and Kominers (2010), the cumulative offer process with choice functions satisfying these conditions produces a stable allocation and is strategy-proof. The former fact, together with Remark 1 and Proposition 1.2, implies that the outcome of the Nested Deferred Acceptance Algorithm is a stable matching in the original model. By Remark 1, we conclude that the NDA mechanism is strategy-proof for doctors.

In order to find an assignment between hospitals and doctors that respect the distributional constraints and regional caps, Kamada and Kojima introduced the Generalized Flexible Deferred Acceptance (GFDA) algorithm.

For each region r fix a quasi-choice rule $\tilde{C}h_r$. The GFDA algorithm proceed as follows

1. Begin with an empty matching, i.e. $\mu_d = \varnothing$ for all $d \in D$.

2. Choose a doctor d arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.

3. Let d apply to the most preferred hospital $\overline{h}$ at H_d among the hospitals that have not rejected d so far. If d is unacceptable to $\overline{h}$, then reject this doctor and go back to step 2. Otherwise, let r be the region such that $\overline{h} \in H_r$ and define vector $\omega = (\omega_h)_h \in H_r$ by

   (a) $\omega_{\overline{h}}$ is the number of doctors currently held at $\overline{h}$ plus one, and

   (b) w_h is the number of doctors currently held at h if $h \neq \overline{h}$,

4. Each hospital $h \in H_r$ considers the new applicant d (if $h = \overline{h}$) and doctors who are temporarily held from the previous step together. It holds its $(\tilde{C}h_r(w))_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to step 2.

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